# While Stability Lasts: A Stochastic Model of Stablecoins

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The 'Black Thursday' crisis in cryptocurrency markets demonstrated deleveraging risks in over-collateralized lending and stablecoins. We develop a stochastic model of over-collateralized stablecoins that helps explain such crises. In our model, the stablecoin supply is decided by speculators who optimize the profitability of a leveraged position while incorporating the forward-looking cost of collateral liquidations, which involves the endogenous price of the stablecoin. We formally characterize regimes that are interpreted as stable and unstable for the stablecoin. We prove bounds on the probabilities of large deviations and quadratic variation in the stable domain and distinctly greater price variance in the unstable domain. The unstable domain can be triggered by large deviations, collapsed expectations, or liquidity problems from deleveraging. We formally characterize a deflationary deleveraging spiral as a submartingale that can cause the system to behave in counterintuitive ways due to liquidity problems in a crisis. These deleveraging spirals, which resemble short squeezes, lead to faster collateral drawdown (and potential shortfalls) and are accompanied by higher price variance, as experienced on Black Thursday. We also demonstrate 'perfect' stability results in idealized settings and discuss mechanisms which could bring realistic settings closer to such idealized stable settings.

# **1 INTRODUCTION**

On March 12, 2020, called 'Black Thursday' during the COVID-19 market panic, cryptocurrency prices dropped ~ 50% in the day.<sup>1</sup> This was accompanied by cascading liquidations on cryptocurrency leverage platforms, including both centralized platforms like exchanges and new decentralized finance (DeFi) platforms that facilitate on-chain over-collateralized lending. Among many events from this day, the story of Maker's stablecoin Dai stands out, which entered a deflationary deleveraging spiral (akin to a short squeeze on Dai). This triggered high volatility of the 'stable' asset and a breakdown of the collateral liquidation process. Due to market illiquidity exacerbated by network congestion, some collateral liquidations were performed at near-zero prices. As a result, the system developed a collateral shortfall, which prompted an emergency response and had to be made up by selling new equity-like tokens to recapitalize [26].

During this time, there was a huge demand for Dai. It became a much riskier and more volatile asset, yet traded at a high premium and fetched lending rates in the mid double digits. Leveraged speculators, who must repurchase Dai in order to deleverage their positions, were exhausting Dai liquidity, driving up the price of Dai and subsequently increasing the cost of future deleveraging (we discuss some further causes that led to market illiquidity in developing the model in the next section). These speculators began to realize that, in these conditions, they face concrete risk that a debt reduction of \$1 could cost a significant premium. Eventually, a new exogenously stable asset—the USD-backed custodial stablecoin USDC–had to be brought in as a new collateral type to stabilize the system [10].

### 1.1 Stablecoins

A stablecoin is a cryptocurrency with added economic structure that aims to stabilize price/purchasing power. For a recent overview of stablecoins, see [5, 21] and the references therein. Stablecoins are meant to bootstrap price stability into cryptocurrencies as a stop-gap measure for adoption. Current projects are either *custodial* and rely on custodians to hold reserve assets off-chain (e.g., \$1 per

<sup>&</sup>lt;sup>1</sup>This occurred while writing up the first draft of this paper.

coin) or *non-custodial* and set up a risk transfer market through on-chain contracts. Non-custodial stablecoins aim to retain the property of reduced counterparty/censorship risk.

Non-custodial stablecoins transfer risk from stablecoin holders to speculators, who hold leveraged collateralized positions in cryptocurrencies.<sup>2</sup> A dynamic deleveraging process balances positions if collateral value deviates too much, as determined by a price feed. This is similar to a tranche structure, in which stablecoins act like senior debt, with the addition of dynamic deleveraging. Two major risks in these stablecoins emerge around market structure collapse and price feed and governance manipulation. In this paper, we focus completely on the market structure risk, assuming that price feeds, governance, and the underlying blockchain perform as expected.<sup>3</sup>

In addition to the COVID-19 panic, the effects of these risks are also witnessed in bitUSD, Steem Dollars, and NuBits, which suffered serious depegging events in 2018 [19], and Terra and Synthetix, which suffered price feed manipulation attacks in 2019 ([40], [39], [37]) and similar manipulations on the bZx lending protocol in 2020 ([32], [33]). Many similar examples of mechanism failures and exploitations occurred through the rest of 2020 (see [21, 43]). Stablecoins currently serve a central role in an increasingly complex decentralized finance environment, involving composability with other DeFi platforms. In addition, many other blockchain assets, such as synthetic and cross-chain assets, rely on the basic mechanism behind stablecoins, which we discuss further in the discussion section.

# 1.2 This paper

In this paper, we construct a stochastic model of over-collateralized stable assets, including noncustodial stablecoins, with an endogenous price (Section 2). The system is based on a speculator who solves an optimization problem accounting for potential returns from leverage as well as potential liquidation costs. The speculator decides the supply of stablecoins secured by its collateral position while considering demand for the stablecoin.

We derive fundamental results about the model, including economic limits to the speculator's behavior, in Section 3. In Section 4 we develop the primary results of the paper: we analytically characterize regions in which the stablecoin can be intepreted as stable (Theorems 1 and 2) and unstable (Theorems 4 and 5), and a region in which a deleveraging spiral occurs that can cause liquidity problems in a crisis (Theorem 3). These deleveraging spirals, which resemble short squeezes, are counterintuitive as they lead to stablecoin price appreciation during times of shock, whereas we might otherwise expect prices to depreciate given the riskier state of the system. Further, this appreciation is detrimental: it leads to faster collateral drawdown, and potentially shortfalls, as more collateral is required to fulfill liquidations and is accompanied by higher price variance.

The context for our analytical results is a model with a single speculator facing imperfectly elastic demand for the stablecoin; however, many of the methods can extend to generalized settings. In Section 5, we consider idealized settings that lead to 'perfect' stability properties. In Appendix A, we consider extensions of the model to generalized design and market settings and consider how results will differ given different model structures. Furthermore, we explore practical applications to realistic settings.

We discuss in Section 6 a seeming contradiction that arises: while the goal is to make decentralized non-custodial stablecoins, these can only be fully stabilized from deleveraging effects by adding uncorrelated assets, which are currently centralized/custodial. This is a consequence of our instability results in Section 4 and, as introduced in Section 5, the absence of a stable region in

<sup>&</sup>lt;sup>2</sup>'Leverage' means that speculators holds >  $1 \times$  initial assets but face new liabilities.

<sup>&</sup>lt;sup>3</sup>Note, however, that blockchain congestion can serve to decrease elasticity in the market structure, which we discuss in the model construction.

idealized settings when underlying asset markets deviate from a submartingale setting. We suggest an alternative: a buffer to dampen deleveraging effects without directly incorporating custodial assets. This buffer works by separating those who are willing to have stablecoins swapped to custodial assets in a crisis (in return for an ongoing yield from option buyers) from those who require full decentralization.

#### 1.3 Relation to Prior Work

While there is a rich literature on related financial instruments, there is limited research directly applicable to stablecoins.

A simple stable asset model is developed in [22] and introduces the concept of deleveraging spirals, which later materialized on Black Thursday. This paper supersedes that model and its results. Whereas the model in [22] doesn't directly account for the actual repurchase price in deleveraging–instead delegating to a risk constraint in the optimization–we set up a stochastic process model in this paper that includes forward-looking liquidation prices in the speculator's optimization. Our analytical results in this paper supersede [22] in the following ways:

- We formally characterize a deleveraging spiral as a submartingale, whereas [22] lacks a formal treatment.
- Stability results in [22] are based on a volatility estimator. We prove stability in terms of probabilities of large deviations and quadratic variation.
- An unstable region is conjectured in [22], backed by simulation. We formally prove distinct price variances in stable and unstable regions.

Option pricing theory is applied in [7] to value tranches in a proposed stablecoin using PDE methods. In doing so, they need the simplifying assumption that payouts (e.g., from liquidations) are exogenously stable, whereas they are actually made in ETH and can cause price feedback effects in the stable asset. In particular, stablecoin holders either hold market risk or are required to re-buy into a reduced stablecoin market following liquidations. This motivates our model to understand stablecoin feedback effects.

[14] analyzes credit risk stemming from collateral type in Maker's stablecoin Dai. [9, 34] model stability in Terra and Celo stablecoins under Brownian motion scenarios in the absence of endogenous market feedback effects that motivate this paper. [20] discuss governance and oracle attack surfaces in non-custodial stablecoins, which is extended with general models in [21] and discussed more generally in decentralized finance as governance extractable value in [23, 43].

[11] discusses stablecoin concepts based on monetary policy and hedging strategies and introduces methods for enhancing liquidity using combinatorial auctions and automated market makers. [24] studied custodial stablecoins and considers the use of hedging techniques to build an asset-backed cryptocurrency. [15] explores the robustness of decentralized lending protocols to shocks and liquidations. [8] explores competition between decentralized lending yields and staking yields in proof-of-stake blockchains. However, these do not model a stablecoin mechanism with endogenous price behavior.

[17] designs a reputation system for crypto-economic protocols to reduce collateral requirements. This does not readily apply to understanding stablecoin collaterals, however, as it requires identification of 'good' behavior and, additionally, stablecoin speculators face leveraged exchange rate bets and will have reason to provide greater than minimal collateral. This additionally motivates our model to understand how liquidation effects affect speculator decisions.

Stablecoins share similarities with currency peg models, e.g., [16, 29]. In these models, the government plays a mechanical market making role to seek stability and is not a player in the game. In contrast, in non-custodial stablecoins, decentralized speculators take the market making

role. They issue/withdraw stablecoins to optimize profits and are not committed to maintaining a peg. In a stablecoin, the best we can hope is that the protocol is well-designed and that the peg is maintained with high probability through incentives. A fully strategic model would be a complicated (and likely intractable) dynamic game.

There are also similarities with collateral and debt security markets and repurchase agreements. These have also experienced unprecedented stress in the COVID-19 market panic, during which even 30-year US government bonds-normally highly liquid-have been difficult to trade [36]. Such debt securities differ from stablecoins in that dollars are borrowed against the collateral as opposed to a new instrument, like a stablecoin, with an endogenous price. These debt security markets do, however, demonstrate that liquidity in the underlying markets can dry up in crises even in highly liquid markets. Stablecoins face this liquidity risk in the underlying market as well as an endogenous price effect on the stable asset.

The problem resembles classical market microstructure models (e.g., [30]); it is a multi-period system with agents subject to leverage constraints that take recurring actions according to their objectives. In contrast, the stablecoin setting has no exogenously stable asset that is efficiently and instantly available. Instead, agents make decisions that endogenously affect the price of the 'stable' asset and affect future incentives.

#### 2 MODEL

Our model is very closely related to Maker's stablecoin Dai [27] as well as newer stablecoins by UMA, Reflexer, and Liquity. We later discuss how it can be adapted to describe other stablecoins such as Synthetix sUSD. The model contains a stablecoin market and two assets: a risky asset (ETH)<sup>4</sup> with exogenous price  $X_t$  and an ETH-collateralized stablecoin STBL with endogenous price  $Z_t$ . The stablecoin market connects stablecoin holders, who seek stability, and speculators, who make leveraged bets backing STBL. The STBL protocol requires the STBL supply to be over-collateralized in ETH by collateral factor  $\beta$ .

In order to focus on the effects of speculator decisions in this paper, we simplify the stablecoin holder demand as exogenous with constant unit price-elasticity. This is equivalent to a fixed STBL demand  $\mathcal{D}$  in dollar terms, though not quantity. We relax this to arbitrary elasticity in Appendix A, including the perfectly elastic case. Note that there is no direct redemption process for stablecoin holders aside from a global settlement/shutdown of the system at par value, which can be triggered by a governance process (see [27]).

From a practical perspective, STBL demand is not elastic, at least short-term, even if it were in principle elastic longer-term. A significant portion of stablecoin supplies are locked in other applications, like lending protocols and lotteries. These applications promise (in some sense) value safety in over-collateralization, but don't guarantee liquidity to withdraw. Additionally, Ethereum transactions cannot be executed in parallel; during volatile times, transactions can be delayed due to congestion, causing timely trades (especially involving transfer to/from centralized exchanges) to fail. This occurs even if, in principle, there is liquidity in these markets. On the other hand, longer-term demand elasticity will naturally depend on the presence of good uncorrelated alternatives.<sup>5</sup>

We focus on the case of a single speculator, though we consider generalizations that accommodate many speculators in Appendix A. The speculator has ETH locked in the system and decides the STBL supply, which represents a liability against its locked collateral. At the start of step t, there are  $\mathcal{L}_{t-1}$  STBL coins in supply. The speculator holds  $N_{t-1}$  ETH and chooses to change the STBL

 $<sup>^{4}</sup>$ We designate the risky collateral asset as ETH for simplicity. In principle, it could be another cryptoasset or even outside of a cryptocurrency setting.

<sup>&</sup>lt;sup>5</sup>From another perspective, a strategic stablecoin holder would take into account expectations about speculator issuance and ability to maintain the price target and expectations about a global settlement. This is outside of our model as formulated.

supply by  $\Delta_t = \mathcal{L}_t - \mathcal{L}_{t-1}$ . If  $\Delta_t > 0$ , the speculator sells new STBL on the market for ETH at the market clearing price  $Z_t$ , which is added to  $N_t$ . If  $\Delta_t < 0$ , the speculator buys STBL on the market, reducing  $N_t$ . The speculator's locked collateral is  $\bar{N}_t$  and may or may not be equivalent to  $N_t$ . Informed by limitations of actual implementations, we develop a particular formulation for the process  $(\bar{N}_t)$  based on  $(N_t)$  in this section, though we discuss ways that this can be generalized in Appendix A.<sup>6</sup> The speculator decides  $\mathcal{L}_t$  by optimizing expected profitability in the next period based on expectations about ETH returns and the cost of collateral liquidation if the collateral factor is breached.

In this way, speculators myopically optimize for the next period. A simplification of our model is a one-off game, which hosts a single period of decision-making before the system is settled in the final period. In this case, the myopic setup is parallel to all major single period games in finance (e.g., [12, 13, 16, 29, 31]). Even here, our results make significant contributions over the existing state of research on stablecoins, describing different system behavior depending on initial conditions in one-off games. The more general multi-period form of our model then describes a dynamic process composed of a series of one-off games with changing initial conditions. Our results also apply more generally to this multi-period setting, where they are stronger than simply a series of the one-off version of the results. Both of these represent significant contributions to stablecoin modeling as there are not better candidates for multi-period models at this point, although we later discuss ideas toward adapting the model into a multi-period control problem.

Given supply and demand, the STBL market clears by setting demand equal to supply in dollar terms. This yields the clearing price  $Z_t = \frac{\mathcal{D}}{\mathcal{L}_t}$ . This clearing equation is related to the quantity theory of money and is similar to the clearing in automated market makers [1] but processed in batch.

#### 2.1 Formal setup

We formalize the model as follows. We define the following parameters:

- $\mathcal{D}$  = STBL demand in dollar value (equivalent to constant unit price-elasticity)
- $\beta$  = STBL collateral factor
- $\alpha \ge 1$  liquidation fee (representing 1+% fee)

The system is composed of the following *processes*:

- $(X_t)_{t \ge 0}$  = exogenous ETH price process
- $\mathcal{L}_t$  = stablecoin supply at time *t* that obeys

$$\mathcal{L}_t = \zeta + L_{t-1} + \Delta_t$$

where  $L_{t-1} > 0$  is the speculator's STBL liabilities from the previous period,  $\Delta_t$  is the speculator's change in liabilities at time *t* (such that  $L_t = L_{t-1} + \Delta_t$ ), and and  $\zeta$  is a real number that modifies circulating supply

- $N_t$  = speculator's ETH position at time *t*, including collateral
- $\bar{N}_t$  = speculator's locked ETH collateral at time *t* (and start of time *t* + 1)
- $(Y_t)_{t \ge 0}$  = speculator's value process
- $Z_t = \frac{\mathcal{D}}{\mathcal{I}_t}$  defines the STBL price process

We take  $(\mathcal{F}_t)_{t\geq 0}$  to be the natural filtration where  $\mathcal{F}_t = \sigma(X_0, \ldots, X_t, \mathcal{L}_0, \ldots, \mathcal{L}_t)$ . The system is driven by the process  $(X_t)$  subject to the speculator's decisions  $\Delta_t$  (equivalently  $L_t$  given  $L_{t-1}$ ).

The parameter  $\zeta$  modifies circulating STBL supply. This could come from an outside amount of STBL not created by the speculator (a positive adjustment), or some STBL could essentially be

<sup>&</sup>lt;sup>6</sup>In principle, the speculator's decision could be extended to deciding  $\bar{N}_t$  in addition to  $L_t$ . Note though that this would make most sense if the speculator's position is further extended to include multiple assets.

locked (a negative adjustment). As formulated, our model applies to a system that can be described with monopolistic agents, or where agents behave similarly (have similar beliefs). With  $\zeta > 0$ , the model becomes similar to having heterogeneous agents. Whereas, in general to do this, we would have to consider both heterogeneous beliefs about the future as well as different  $\zeta s$ , which together would be untreatable in this paper,  $\zeta$  provides a way to aggregate these various effects in a simpler model. In particular, we suggest a positive  $\zeta$  may make numerical results more applicable to real settings.

To simplify the exposition of analytical results going forward, we simplify to the case that  $\beta = \frac{3}{2}$  (the collateral factor used in Maker's Dai stablecoin) and  $\zeta = 0$ . Note that under these conditions, and in the remainder of the paper, we use  $L_t$  and  $\mathcal{L}_t$  interchangeably. However, similar analytical results will extend to the general setting, as we discuss in Appendix A, and we will note any deviations where  $\zeta > 0$  qualitatively changes results.

#### 2.2 Collateral constraint

The collateral constraint requires the collateral locked in the system to be  $\geq$  a factor of  $\beta$  times by liabilities. It applies in both a pre-decision and post-decision sense. The *pre-decision* version determines when a liquidation occurs: a liquidation is triggered at the start of time *t* if

$$N_{t-1}X_t < \beta L_{t-1}$$

The post-decision version constrains the speculator's decision-making, limiting  $L_t$  such that

$$N_t X_t < \beta L_t.$$

# **2.3** Speculator decides $\Delta_t$

We assume the speculator is risk-neutral and optimizes its next-period expected value, taking into account expectations around liquidations. Note that the assumption of risk neutrality can be removed by instead applying an appropriate utility function–in some reasonable cases the results will still hold. Its value at time t is its equity at the start of period (pre-decision), given by

$$Y_t = N_{t-1}X_t - L_{t-1}$$
 – liquidation effect

A liquidation effect is triggered as outlined in a following subsection.

The speculator treats  $L_t$  at face value in the optimization for a mix of myopic and long-term reasons. The risk of STBL market price effects, such as deleveraging spirals, is already accounted for in the liquidation cost component. As long as speculators can survive (e.g., if they aren't completely liquidated), they can expect to dispose of liabilities at face value longer-term when markets are liquid. The protocol designs also add a precedent for treating liabilities at face value: it is treated in this way in the collateral constraint and in the event of global settlement of the system, which could be triggered at any time (and which would occur in the final period of the one-off version).

Note that  $N_t$  is a function of the decision variable  $\Delta_t$ , and recall  $L_t = L_{t-1} + \Delta_t$ . The speculator decides  $\Delta_t$  (equivalently  $L_t$  given  $L_{t-1}$ ) to optimize next-period expected value subject to the post-decision collateral constraint in the current period:

$$\max_{\Delta_t} \qquad \mathbb{E}[Y_{t+1}|\mathcal{F}_t]$$
  
s.t.  $\bar{N}_t X_t \ge \beta L_t$ 

# 2.4 Speculator's collateral at stake

We consider that the speculator decides on a level of participation as a component of their entire portfolio. This takes place in a separate optimization problem outside the scope of this model (although we discuss how it could be extended later). The speculator's level of participation amounts

to the initial collateral at the start of our model-for simplicity, we say this also includes any amount they have decided beforehand may be accessible to top up collateral later. The speculator's behavior in our model amounts to maximizing the expected value of this component of their portfolio. On the other hand, if this were the speculator's entire portfolio, we note that the story may be different-e.g., they may want to maximize expected log values as in the Kelly criterion and would probably choose to participate differently, as is common in problems of leverage if the whole portfolio is at stake.

We take the speculator's collateral at stake at the start of time t + 1 to be  $\bar{N}_t = N_{t-1}$  minus any collateral liquidation that happens at time t. This is consistent with the speculator's collateral being blocked: it cannot be used to repurchase STBL in the same step. This means that the speculator (1) has an outside amount (or is able to borrow) to repurchase STBL if  $\Delta_t < 0$  and then alter repays this from unlocking collateral and (2) can't post proceeds of new STBL issuance ( $\Delta_t > 0$ ) as collateral within the same step.

While there are settings in which we could alternatively use  $N_t$  as the collateral at stake at the start of t + 1 (e.g., if flash loans are used), the choice of  $N_{t-1}$  additionally leads to a simpler exposition of results as it decouples the collateral from the decision variable. This said, the general methods and results would extend into the setting with  $N_t$  collateral, as we discuss in Appendix A.

#### 2.5 Collateral liquidation mechanics

In time t + 1, the pre-decision collateral constraint is  $\bar{N}_t X_{t+1} \ge \beta L_t$ . If this is breached, then the speculator's collateral is partially liquidated, if possible, to repurchase an amount  $\ell_{t+1} > 0$  of STBL. In real protocols, liquidation amounts are automated by an algorithm and will inherently be first order estimates of the amount needed to rebalance the debt position as the algorithm will not be able to know the actual market structure and price impact. For instance, liquidations in Maker and Compound release a certain amount of debt to be repaid, and unlock a corresponding amount of collateral that an arbitrager can use to rebalance the debt position (both decided algorithmically in Compound and Maker's previous version of Dai, and the latter decided through auction in Maker's newer version of Dai). Consistent with these protocols, we set the amount of debt that needs to be repaid in a liquidation to be  $\ell_{t+1}$  of STBL such that post liquidation  $\bar{N}_t X_{t+1} - \ell_{t+1} = \beta(L_t - \ell_{t+1})$ . With  $\beta = \frac{3}{2}$ , this amount is

$$\ell_{t+1} = \frac{\beta L_t - \bar{N}_t X_{t+1}}{\beta - 1} = 3L_t - 2\bar{N}_t X_{t+1}.$$

Other liquidation algorithms could also be considered and would lead to similar qualitative effects.

In a time step with a liquidation, the liquidation forces an upper bound  $\Delta_{t+1} \leq -\ell_{t+1}$  as this amount would, in the real protocol, be unlocked for arbitrageurs. But the speculator could choose to repurchase more STBL to further reduce leverage. The repurchase of  $\ell_{t+1}$  through the liquidation mechanism is subject to a liquidation fee multiple  $\alpha \geq 1$ -i.e., the effective repurchase price is  $\alpha \times$ the STBL market price. The purpose of this fee is that, in real stablecoin systems, these liquidations are performed by arbitrageurs who capture this fee.

Notice that the STBL market price will itself be affected by liquidations. Depending on market impact, which the algorithms can only observe sequentially, the liquidation may be insufficient to fully rebalance the debt position back to the collateral constraint. If this occurs, then the issue will be taken into account with further liquidations in subsequent time steps. The parameter  $\beta$  in real systems is intended to provide safety in such events so that the system does not become under-collateralized.

Two thresholds are relevant at time *t* for calculating expectations of a liquidation effect at time t + 1. These are non-time-dependent functions of the random variable  $L_t$ :

0.7

$$b(L_t) := \frac{\beta L_t}{\bar{N}_t}$$
$$c(L_t) := \frac{1}{2\bar{N}_t} \left( \sqrt{\alpha^2 \mathcal{D}^2 + 4\alpha \mathcal{D}L_t + L_t^2} - \alpha \mathcal{D} + L_t \right)$$

The threshold  $b(L_t)$  gives the highest t + 1 ETH price that breaches the collateral constraint while the threshold  $c(L_t)$  gives the t + 1 ETH price that consumes the entirety of the speculator's locked collateral in a liquidation repurchase due to the effect on STBL repurchase price.<sup>7</sup> Below this level, the speculator cannot meet the collateral demand even by liquidating everything. The formulation of  $b(L_t)$  follows directly from the collateral constraint; the formulation of  $c(L_t)$  follows from equating the repurchase cost of liquidation  $\ell_{t+1}$  to  $\bar{N}_t X_{t+1}$  and solving for  $X_{t+1}$ .

If  $c(L_t) \leq X_{t+1} \leq b(L_t)$ , then the liquidation effect is  $\ell_{t+1} - \ell_{t+1} \frac{\mathcal{D}}{\mathcal{L}_t - \ell_{t+1}} \alpha$ . This represents a repurchase of  $\ell_{t+1}$  STBL (reducing collateral by the repurchase price  $\frac{\mathcal{D}}{\mathcal{L}_t - \ell_{t+1}}$  with liquidation fee factor  $\alpha$ ) and subsequent reduction of the speculator's liabilities by the  $\ell_{t+1}$ . The variables  $\mathcal{L}_{t+1}$  and  $N_t$  are affected similarly.<sup>8</sup> If  $X_{t+1} < c(L_t)$ , then the speculator's collateral position is zeroed out in the liquidation. We define the corresponding events

$$A_t = \{X_{t+1} \ge b(L_t)\}$$
$$B_t = \{c(L_t) \le X_{t+1} < b(L_t)\}.$$

#### 2.6 System of random variables

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Putting all the pieces together, we have the following system of random variables driven by the random process  $(X_t)$ :

$$\begin{split} \chi_{t} \\ Y_{t+1} &= \frac{\Delta_{t} \mathcal{D} X_{t+1}}{\mathcal{L}_{t} X_{t}} + (\bar{N}_{t} X_{t+1} - L_{t}) \, \mathbb{1}_{A_{t} \cup B_{t}} + \mathbb{1}_{B_{t}} (3L_{t} - 2\bar{N}_{t} X_{t+1}) \Big( 1 - \frac{\alpha \mathcal{D}}{2\bar{N}_{t} X_{t+1} - 2L_{t}} \Big) \\ \Delta_{t}^{*} &= \begin{cases} \min \Big( \arg \max_{\Delta_{t}} \mathbb{E}[Y_{t+1} | \mathcal{F}_{t}], \frac{\bar{N}_{t-1} X_{t}}{\beta} - L_{t-1} \Big) & \text{if } X_{t} \geq \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \\ \min \Big( \arg \max_{\Delta_{t}} \mathbb{E}[Y_{t+1} | \mathcal{F}_{t}], -(3\mathcal{L}_{t-1} - 2\bar{N}_{t-1} X_{t}) \Big) & \text{if } X_{t} < \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \\ \mathcal{L}_{t} &= \mathcal{L}_{t-1} + \Delta_{t}^{*} \\ N_{t} &= \begin{cases} N_{t-1} + \Delta_{t}^{*} \frac{Z_{t}}{X_{t}} & \text{if } X_{t} \geq \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \\ N_{t-1} + \frac{Z_{t}}{X_{t}} (\Delta_{t} + (1 - \alpha)(3\mathcal{L}_{t-1} - 2\bar{N}_{t-1} X_{t})) & \text{if } X_{t} < \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \\ \end{cases} \\ \bar{N}_{t} &= \begin{cases} N_{t-1} & \text{if } X_{t} \geq \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \\ N_{t-1} - \alpha(3\mathcal{L}_{t-1} - 2\bar{N}_{t-1} X_{t}) & \text{if } X_{t} < \frac{\beta L_{t-1}}{\bar{N}_{t-1}} \\ Z_{t} &= \frac{\mathcal{D}}{\mathcal{L}_{t}}. \end{cases} \end{split}$$

 $<sup>^{7}</sup>$ Note that the probability of a large deviation like this is not zero. For instance, it could represent the possibility of a contentious hard fork that splits ETH value.

<sup>&</sup>lt;sup>8</sup>Note that  $N_t$  is affected because this is the locked collateral at time t + 1. Alternatively, working with  $N_{t+1}$  as locked collateral, we would update  $N_{t+1}$ .

In the above, the first case for  $\Delta_t^*$  comes from maximizing expected value subject to the postdecision collateral constraint while the second cases for  $\Delta_t^*$ ,  $N_t$ , and  $\bar{N}_t$  apply the liquidation effects that occur during time *t*.

# **3 FOUNDATIONAL RESULTS**

In this section, we derive foundational results about the model that we will use to prove the primary results of the paper in the next section.

### 3.1 Assumptions

We begin by defining the assumptions we will use in the rest of the paper.

ASSUMPTION 1.  $(X_t)$  is a submartingale with respect to  $(\mathcal{F}_t)$  and is independent from  $(\mathcal{L}_t)$  and  $(N_t)$ .

A submartingale is a stochastic process in which the expected future value, conditioned on all prior values, is greater than or equal to the current value. Note that the submartingale assumption can be relaxed somewhat while preserving some results. It is useful, though not necessarily critical, in our proof of problem concavity. However, the results are most meaningful in a setting like a submartingale, which always provides a fundamental reason that a speculator might desire leverage. In such a setting, it is *conceivable* that the stablecoin could maintain a dollar peg, whereas in long periods of negative expected returns, the stablecoin concept falls apart as no speculators will want to participate. As noted in the introduciton, such a deviation from the submartingale setting appears to have occurred in March 2020. In Section 5, we elaborate how the concept falls apart in such negative settings, even given otherwise perfect market structure.

Also note that the submartingale differences need not be independent for most results. In the Appendix, we further consider ways in which independence of  $(X_t)$  and  $(\mathcal{L}_t)$  can be relaxed.

ASSUMPTION 2. Each  $X_{t+1}$  has a conditional probability distribution given  $\mathcal{F}_t$ , which admits a density function  $f_t$  that is a.s. continuous.

Equivalently, we consider the process in terms of returns  $R_t$ , where  $X_{t+1} = X_t R_{t+1}$ . Conditioned on  $\mathcal{F}_t$ , then  $R_{t+1}$  admits density function  $g_t$ . In the i.i.d. setting for  $(R_t)$ , the time dependence can be dropped. As noted above, for most results, we do not need to assume i.i.d.

Assumption 3. There is some upper bound  $r \ge \sup_n \mathbb{E}[R_n | \mathcal{F}_{n-1}]$ .

The next assumption is needed to interchange derivative and integration operators in the improper setting. Note that it also translates to an upper bound on  $\mathcal{L}_t$  and a lower bound on  $N_{t-1}$ .

ASSUMPTION 4. There is some upper bound  $u \ge c(L_t)$  for all  $L_t$ .

The next assumption bounds the STBL price away from singularity. As discussed previously, it can be avoided under an alternative formulation of the model.

Assumption 5.  $\mathcal{L}_t \geq v > 0$  for some v.

The next assumption simplifies repurchase considerations. It is reasonable given a reasonable bound r on expected returns.

ASSUMPTION 6. The liquidation premium factor  $\alpha$  is sufficiently high that the repurchase price in a liquidation is a.s. > 1.

The next assumption translates to a reasonable condition on *X* distributions considering  $b(L_t)$  is increasingly linearly whereas  $c(L_t)$  decreases with  $L_t$ .

Assumption 7.  $\mathbb{P}(B_t | \mathcal{F}_t) = \mathbb{P}\left(c(L_t) \leq X_{t+1} \leq b(L_t) | \mathcal{F}_t\right)$  is increasing in  $L_t$ .

Define  $\psi(\mathcal{L}_t) := \mathbb{E}[Y_{t+1}|\mathcal{F}_t]$ . Note that  $\psi$  could have a subscript *t*, or equivalently other time *t* inputs  $(\bar{N}_t, X_t, g_t)$ , but we relax notation as we only use it in the context of time *t*. The next assumption ensures that  $\psi$  is concave in  $\mathcal{L}_t$ , a result that we prove in Prop. 1.

Assumption 8. 
$$\frac{\alpha \mathcal{D}Nc_t}{2(Nc_t - \mathcal{L}_t)^2} \leq 2$$
 (note  $\mathcal{L}_t \geq \frac{27}{46} \alpha \mathcal{D}$  is sufficient)

Additionally, the next assumption ensures that  $\psi$  is *strictly* concave in  $\mathcal{L}_t$ , which we also prove in Prop. 1. Notice that this means that *either* the submartingale inequality is strict at time *t* or there is non-zero probability that a liquidation is triggered in the next step. Given that the latter is certainly reasonable, this assumption is not much stronger than the basic submartingale assumption.

Assumption 9. Either 
$$\mathbb{E}[R_{t+1}|\mathcal{F}_t] > 0$$
 or  $\mathbb{P}(B_t|\mathcal{F}_t) = \mathbb{P}\left(c(L_t) \le X_{t+1} \le b(L_t)|\mathcal{F}_t\right) > 0$ .

While strict concavity of  $\psi$  is not necessary for all results, it does simplify the analysis considerably. More generally, concavity of  $\psi$  could reasonably be expected in many settings, and so the assumptions can probably be relaxed. Informally, reasonable distributions for  $X_t$  will have concentration about the center. In this case, moving  $\Delta_t$  in the positive direction, expected liabilities increase faster than revenue from new STBL issuance. Moving  $\Delta_t$  in the negative direction, the cost to buyback grows faster than the decrease in expected liabilities.

# 3.2 Concavity and scale invariance

Our first result is to prove that  $\psi(\mathcal{L}_t)$  is concave in  $\mathcal{L}_t$ .

PROP. 1. Given Assumptions 1-8,  $\psi(\mathcal{L}_t) := \mathbb{E}[Y_{t+1}|\mathcal{F}_t]$  is concave in  $\mathcal{L}_t$ . Further, given additional Assumption 9,  $\psi(\mathcal{L}_t)$  is strictly concave in  $\mathcal{L}_t$ .

#### [Link to Proof]

In deriving some results, it will be useful to make assumptions about the scale of the system. The next result shows that results about  $Z_t$  should translate to differently scaled systems, validating that such results will describe the STBL price process more generally. In the following, we define h to output  $\mathcal{L}_t$  as a function of the system state.

**PROP.** 2. Consider a system setup  $(L_{t-1}, \mathcal{D}, N_{t-1})$  with ETH price process  $(X_t)$ . For  $\gamma > 0$ ,

$$h(\gamma L_{t-1}, \gamma \mathcal{D}, \gamma N_{t-1}, X_t) = \gamma h(L_{t-1}, \mathcal{D}, N_{t-1}, X_t)$$
$$h(L_{t-1}, \mathcal{D}, \frac{1}{\gamma} N_{t-1}, \gamma X_t) = h(L_{t-1}, \mathcal{D}, N_{t-1}, X_t)$$

As a result, the STBL price process  $(Z_t)$  is equivalent across these system rescalings.

# [Link to Proof]

Under these conditions, we can interchange derivative and integration operators in  $\frac{\partial \psi}{\partial \mathcal{L}_t}$  according to Leibniz integral rules (a variation of dominated convergence theorems). The speculator's choice of  $\mathcal{L}_t$  will fulfill the first order condition of  $\frac{\partial \psi}{\partial \mathcal{L}_t} = 0$ . From concavity, we can then conclude that the speculator chooses to increase the STBL supply when  $\frac{\partial \psi}{\partial \mathcal{L}_t}(\mathcal{L}_{t-1}) > 0$  and to decrease the STBL supply when  $\frac{\partial \psi}{\partial \mathcal{L}_t}(\mathcal{L}_{t-1}) > 0$ .

Note that we can derive sufficient conditions for these events using Lemma 2 from the Appendix. Such conditions can be useful as concrete interpretations of the events and can be checked against incoming data. That said, these general sufficient conditions are far from necessary if we are given additional information about the return distributions.

## 3.3 Economic limits to speculator behavior

We now present some fundamental results that bound the speculator's decision-making. These results will be useful in developing the primary results of the paper in the next section. The next result introduces a lower bound to the speculator's STBL supply decision that arises from the fundamental price impact of repurchasing STBL.

**PROP. 3.** Suppose the pre-decision collateral constraint is met at time t. There is a computable lower bound to  $\Delta_t$ .

We can interpret the lower bound in terms of a balance sheet constraint describing when the speculator's ETH position is exhausted in a repurchase. We give the specific bound in the proof but note that it is not especially useful on its own. Given information about the returns distribution and the level of current collateral and considering  $\frac{\partial \psi}{\partial \mathcal{L}_t}$ , much better bounds are possible. Note that if  $\zeta > 0$  is high enough, the lower bound may be the speculator's entire debt position, which would be expected in a liquid environment with heterogeneous agents.

#### [Link to Proof]

The next result provides a useful upper bound to the speculator decision  $\mathcal{L}_t$ . The result is derived from incentives to issue STBL. Intuitively, it says that if supply is below this bound, then in some sense a marginal speculator may see a profitable opportunity to expand supply. It's simply not profitable to issue more STBL than this bound. This doesn't mean that the speculator decides to achieve the bound, however, as it underestimates the liquidation costs that the speculator might face.<sup>9</sup> Notice that the bound is strongest when we have  $\kappa \sim 1$ .

**PROP.** 4. Suppose either of the following hold for given  $\kappa$ :

• 
$$\int_{\frac{c(L_t)}{X_t}}^{\frac{b(L_t)}{X_t}} \left(3 - \frac{\alpha \mathcal{D}\bar{N}_t X_t z}{2(\bar{N}_t X_t z - \mathcal{L}_t)^2}\right) g_t(z) dz \le 0 \text{ and } \mathbb{P}(A_t \cup B_t | \mathcal{F}_t) \ge \kappa^{-1} > 0$$
  
• 
$$1 \ge \mathbb{P}(A_t | \mathcal{F}_t) - 2 \mathbb{P}(B_t | \mathcal{F}_t) \ge \kappa^{-1} > 0$$

Then  $\mathcal{L}_t \leq \sqrt{\kappa \mathcal{L}_{t-1} \mathcal{D} \mathbb{E}[X_{t+1} | \mathcal{F}_t] / X_t}$ 

#### [Link to Proof].

The first condition comes from the derivative of the expected liquidation effect with respect to  $\mathcal{L}_t$  taking  $\beta = \frac{3}{2}$ . The integrand can be interpreted as the effective leverage change in a given liquidation. Notice that this is < 0 evaluated at  $b(L_t)$  (small liquidations effectively reduce leverage) whereas it is > 0 evaluated at  $c(L_t)$  (in very large liquidations, leverage reduction may not be effective due to effect on repurchase price). The integral condition then says that, in expectation, liquidations effectively reduce leverage. This is a generally reasonable assumption given a starting state of sufficient over-collateralization, since reasonable distributions of  $X_{t+1}$  will place most mass in the integral around  $b(L_t)$  as opposed to  $c(L_t)$ , which is a tail event.

The second (alternative) condition says that the probability of having a liquidation is sufficiently smaller than not having a liquidation.

This result holds if *either* of the two conditions hold, both of which could be checked in datadriven modeling. We will formalize an assumption like the first condition in the next section. Note, however, that similar results going forward could be derived instead using a variation on the second condition.

<sup>&</sup>lt;sup>9</sup>Note that the model as formulated does not incorporate an interest rate paid by the speculator on issued STBL (the 'stability fee' in Dai). Additionally, it does not incorporate a possible yield if the speculator creates STBL to lend on a lending platform as opposed to selling on the market. Under either of these extensions, Prop. 4 would change by an appropriate factor.

#### STABLE AND UNSTABLE DOMAINS 4

The primary results of the paper characterize regions in which the stablecoin price process can be interpreted as 'stable' and 'unstable'. In this section, we derive these results for the given model of a single speculator facing imperfectly elastic demand for STBL. In the next section, we consider generalizations of the model and how these results will differ given different design and market structures.

#### **Domain barriers/Stopped processes** 4.1

We first establish results in terms of barriers. While the stablecoin process is within certain barriers, we prove that it behaves in ways that are interpretable as 'stable' and 'unstable'. These barriers are generally stopping times, and we proceed by considering the stopped processes.

Assume that in the initial condition,  $\mathbb{E}\left|\frac{1}{\mathcal{L}_1}|\mathcal{F}_0\right| \leq \frac{1}{\mathcal{L}_0}$ . We define the following stopping times:

- *τ* is the hitting time of E [ <sup>1</sup>/<sub>L<sub>t+1</sub></sub> | *F<sub>t</sub>*] > <sup>1</sup>/<sub>L<sub>t</sub></sub>, *T<sub>m</sub>* is the hitting time of *Z<sub>t</sub>* > *m*, for *m* ≥ *Z*<sub>0</sub>,
- $S_1$  is the hitting time of  $\mathbb{E}[\mathcal{L}_{t+1}|\mathcal{F}_t] < \mathcal{L}_t$ ,
- $S_2$  is the hitting time of  $\mathbb{E}[\mathcal{L}_{t+1}|\mathcal{F}_t] \ge \mathcal{L}_t$  such that  $S_2 > S_1$ .

As we will see, while the stablecoin mechanism is working as intended, we generally expect the STBL supply to increase (equivalently in this setting, the STBL price to decrease, though in slow and bounded way). With this context in mind,  $\tau$  represents the first time we *expect* the STBL price to increase. Notice that this is an expectation of reciprocal of supply, a convex function, and so through Jensen's inequality, this is weaker than expecting the speculator to deleverage/reduce supply. It will be influenced heavily by the tails possibilities. In particular, we have  $\tau \leq S_1$ .

Note that the expectations of the process are not necessarily the same as the actual movements of the process:  $\tau$  does not necessarily correspond to the first time the process actually increases in price. We track this with  $T_m$ , the time the STBL price breaches a given level above  $Z_0$ , which may be before or after  $\tau$ .

The stopping times  $S_1$  and  $S_2$  track when expectations about STBL supply change. These can be equivalently stated (and calculated in a data-driven model) based on expectations about the derivative of  $\mathbb{E}[Y_{t+2}|\mathcal{F}_t]$  with respect to  $\mathcal{L}_{t+1}$  evaluated at  $\mathcal{L}_t$ , similarly to the discussion from the previous section on concavity.

Before proceeding, we formalize stopped versions of assumptions in Prop. 4. The interpretation of these assumptions is the same as discussed in the previous section. Note that the results going forward could also apply more generally subject to additional stopping times embedding these assumptions. For notational simplicity, we just present the results subject to the stopping times already defined with the assumptions given.

Assumption 10. For  $t \leq \tau$ ,  $\mathbb{P}(A_t \cup B_t | \mathcal{F}_t) = \mathbb{P}(X_{t+1} \geq c(L_t) | \mathcal{F}_t) \geq \kappa^{-1} > 0$ .

Assumption 11. For 
$$t \leq \tau$$
,  $\int_{\frac{c(L_t)}{X_t}}^{\frac{b(L_t)}{X_t}} \left(3 - \frac{\alpha \mathcal{D}\bar{N}_t X_t z}{2(\bar{N}_t X_t z - \mathcal{L}_t)^2}\right) g_t(z) dz \leq 0$ 

Notice that  $\kappa$  will be > 1 but ~ 1 as  $X < c(L_t)$  is a low probability event.

Recall that the STBL price  $Z_t$  is a function of collateral value, expectations about ETH returns, and expectations of liquidation costs (related to tail risks). These factors go into the speculator's supply decision, which goes into  $Z_t$ . Going forward, we will explore how changes in these affect the STBL price process.

#### 4.2 'Stable' domain

Subject to the barriers  $\tau$  and  $T_m$ , the stablecoin process can be interpreted as stable in the following ways. In this domain, we derive bounds on large price movements and quadratic variation. We show below that for realistic values of parameters, the bounds are sufficiently powerful in practice.

Our first result bounds  $Z_t$  under the condition  $T_{Z_0} > \tau$ . Conditioned on this, the price is contained within small variation–e.g., consider  $Z_0 = 1$  and consider  $\frac{1}{\kappa r} \sim 1$ .

PROP. 5. Let  $r := \sup_t \frac{\mathbb{E}[X_{t+1}]}{X_t}$ . If  $T_{Z_0} > \tau$ , then

$$Z_0 \geq Z_{t \wedge \tau} \geq \sqrt{\frac{\mathcal{D}}{\kappa \mathcal{L}_{t \wedge \tau - 1} r}} \geq \frac{\mathcal{D}}{(\kappa \mathcal{D} r)^{\frac{2^t - 1}{2^t}} \mathcal{L}_0^{\frac{1}{2^t}}}$$

Furthermore for any t,  $\mathcal{L}_{t\wedge\tau} \leq \kappa \mathcal{D}r$  and  $Z_{t\wedge\tau} \geq \frac{1}{\kappa r}$ .

#### [Link to Proof]

Notice, however, that the condition  $T_{Z_0} > \tau$  introduces dependence on future events. As such, we can't conclude with the information at time *t* that the *t* + 1 price is bounded in this way.

However, we can bound our expectations on the t + 1 price given the information at time t ( $\mathcal{F}_t$ ). This approach relies on the fact that the versions of the process behaves nicely as submartingales in the stopped setting.

**PROP.** 6.  $(\mathcal{L}_{t\wedge\tau})$  is a submartingale bounded above and  $(Z_{t\wedge\tau})$  is a supermartingale bounded below. Thus they converge a.s.

#### [Link to Proof]

An immediate bound on expected price comes from the fact that stopped version of  $Z_t$  is a supermartingale. This is the first result of the next proposition. Additionally, with a stronger assumption on  $(X_t)$  that conditional expectation of returns is non-decreasing within the domain barriers, we can bound the expected price further.

**PROP.** 7. The process  $(Z_{t \wedge \tau \wedge T_{Z_0}})$  is bounded in expectation by

$$Z_0 \geq \mathbb{E}[Z_{t \wedge \tau \wedge T_{Z_0}}] \geq \frac{1}{\kappa r}.$$

Further, assuming that for  $t < \tau$ ,  $(\mathbb{E}[R_{t+1}|\mathcal{F}_t])$  is non-decreasing, then for  $t \leq \tau$ ,

$$Z_{t-1} \geq \mathbb{E}[Z_{t\wedge\tau}|\mathcal{F}_{t-1}] \geq \sqrt{\frac{\mathcal{D}}{\kappa \mathcal{L}_{t-1} \mathbb{E}[R_t|\mathcal{F}_{t-1}]}}$$

[Link to Proof]

Going forward, we will work with a variation on the price process

$$Z'_t := |m - Z_t|$$
 for given  $m \ge Z_0$ .

Using m = 1, this has concrete interpretation as the absolute price deviation from the stablecoin peg. The stopped version of this process has the useful property of being a non-negative submartingale. In addition, ( $Z'_t$ ) shares similar large deviation and quadratic variation properties with ( $Z_t$ ), which we explore in the remainder of this subsection.

LEMMA 1. The stopped process  $(Z'_{t \land \tau \land T_m})$  is a non-negative submartingale.

[Link to Proof]

We define the maximum process over some process  $(\theta_t)$  as  $\theta_N^* = \max_{t \le N} |\Theta_t|$ . The next result bounds the expected maximum of the deviation process  $(Z_t)$ .

PROP. 8. Suppose  $m \ge Z_0$ . Denote  $E := \mathbb{E}[Z_{\tau \wedge T_m} - m | Z_{\tau \wedge T_m} > m]$ . Suppose any one of the following conditions holds:

- $\frac{1}{\kappa r} > m \text{ and } E > \frac{1}{\kappa r} m$   $\frac{1}{\kappa r} = m \text{ and } E > 0$   $\frac{1}{\kappa r} < m \text{ and } E \ge 0$

Then

 $\mathbb{E}[Z_{\tau \wedge T_m}^{\prime*}] \le 2\left(m - \frac{1}{\kappa r}\right).$ 

# [Link to Proof]

The value  $(m - \frac{1}{\kappa r})$  describes the range of the domain considered. Prior to  $T_m$ , we know that the price falls in this range. The nontrivial part is describing what happens at the stopping time as it *exceeds* this range if the stop is triggered by  $T_m$ . The value E is the expected deviation at the stopping time given that  $T_m$  triggers the stop. By definition, E > 0. Given reasonable  $\kappa$ , r, and m, the condition for Prop. 8 is satisfied quite broadly. For instance, the concrete instance with m = 1 is satisfied since  $\frac{1}{\kappa r} < 1$  taking into account the above discussion on  $\kappa$ .

Notice that the analysis for the proof can lead to better bounds if we have more information about *E* or  $p := \mathbb{P}(Z_{\tau \wedge T_m} \leq m)$ , e.g., by incorporating information from other results above or from knowledge about the distributions of  $(X_t)$ , such as from historical data. Additionally, the analysis can be used to bound either *E* or *p* given bounds on the other.

We now state the first main results of the paper. Our next result applies Doob's inequality to bound the probability of large deviations in the stopped process.

THEOREM 1. For  $m \ge Z_0$  and  $\epsilon > 0$ ,

$$\mathbb{P}\left(\max_{n\leq\tau\wedge T_m}Z'_n>\epsilon\right)\leq 2\epsilon^{-1}\left(m-\frac{1}{\kappa r}\right).$$

## [Link to Proof]

The result can be pretty powerful. Consider the concrete case of m = 1, in which case  $Z'_t$  describes the deviation from the peg, and take (arguably reasonable)  $\kappa^{-1} = 0.999$  (99.9% chance  $X_t$  won't drop below  $c(L_t)$  and r annualized as 1.5 (daily r = 1.0011). Then the probability that the stablecoin deviates from the peg by more than 0.1 is  $\mathbb{P}(Z_{\tau \wedge T_1}^{\prime*} > 0.1) \leq 0.042$ .

Our next result derives from a form of Burkholder's inequality that applies to non-negative submartingales. We define the quadratic variation of  $(Z'_t)$  by

$$[Z']_t := \sum_{k=1}^t (Z'_k - Z'_{k-1})^2$$

The quadratic variation is a stochastic process that measures how spread out the underlying process is. Its expectation at time t is related to the variance at that time, supposing variance is defined-in particular, they are equal if the underlying process is a martingale. The result bounds the probability of large quadratic variation in the stopped process. In essence, with high probability, the quadratic variation can't be too far away from the expected maximum.

THEOREM 2. Suppose  $m \ge Z_0$  and  $\epsilon > 0$ . Then

$$\mathbb{P}\left(\sqrt{[Z']_{\tau \wedge T_m}} > \epsilon\right) \le 6\epsilon^{-1} \left(m - \frac{1}{\kappa r}\right)$$
[Link to Proof]

This result is also pretty powerful. Considering the same setting as above, we have  $\mathbb{P}(\sqrt{[Z']_{\tau \wedge T_1}} > 0.1) \leq 0.127$  in the stable domain.

Bounds on the expectation of quadratic variation can also be obtained using a more classical form of Burkholder's inequality, albeit with stronger assumptions. We develop this idea in the next remark.

REMARK 1. There is an additional form of Burkholder's inequality that extends to non-negative submartingales. If we are additionally given a useful bound on  $\mathbb{E}\left[\left(Z'_{\tau \wedge T_m}\right)^p\right]$  for some  $1 (for instance, if we have some distribution assumptions on <math>(X_t)$ ), then we can apply Lemma 3.1 in [6] to derive the following bound on quadratic variation expectations:

$$\mathbb{E}\left[\left([Z']_{\tau\wedge T_m}\right)^p\right]^{\frac{1}{p}} \leq \frac{9p^{\frac{1}{2}}}{1-p^{-1}} \mathbb{E}\left[\left(Z'_{\tau\wedge T_m}\right)^p\right]^{\frac{1}{p}}.$$

There is a lot of research on obtaining the best constants/bounds in Burkholder's inequality, which may be able to tighten the bound.

Note that the classical two-sided Burkholder inequality may not extend to non-negative submartinagales. In general, only the first half of the Burkholder inequality (bounding expectations about quadratic variation by the maximum) extends to this setting and only for 1 . This contrastswith Prop. 2, where we can derive results about probability of large quadratic variation of non-negativesubmartingales for the <math>p = 1 case. From a practical point of view, this may be good enough.

Notice that with an effective bound on the expectation of quadratic variation (QV) of the entire stable process, we have by law of large numbers

$$\frac{QV}{n} \to 0 \text{ as } n \to \infty.$$

So the longer the process is stable, the smaller the variability.

As we've characterized this 'stable' domain based on  $\tau$  and  $T_m$ , an exit from this region corresponds to either a change in expectations ( $\tau$ ) or a large deviation event ( $T_m$ ). In actual applications, we will know when these stopping times arrive (or will at least have good measures of it, when hard to directly observe). These could be used by system stakeholders as indicators that the local regime is changing. Statistical analysis on historical data could also predict how likely we are to see such indicators in coming steps.

# 4.3 'Unstable' domain

We now characterize how the stablecoin can be interpreted as unstable outside of the barriers described above. The intuition here is that the speculator's position is nearer to  $c(L_t)$  and  $b(L_t)$ , and so expected costs of liquidation increase and are more sensitive to the threshold proximity, in addition to being driven by the volatile process  $(X_t)$ . The remaining results in this section characterize a deflationary regime that is connected with instability in terms of forward-looking variance of stablecoin prices and large deviations. In this regime, we observe deleveraging spirals, which resemble short squeezes, and are counterintuitive as they lead to stablecoin price appreciation during times of collateral shock and lead to faster collateral drawdown.

Our next result characterizes a deflationary regime defined by stopping times  $S_1$  and  $S_2$ . In such a setting, an opposite behavior occurs compared to the stable region:  $(Z_t)$  behaves as a submartingale, tending to increase in price. This behavior is caused by *deleveraging spirals*, akin to short squeezes in which liquidations exhaust stablecoin liquidity and lead to stablecoin price increases and exacerbate collateral drawdown.



Fig. 1. Illustration of deleveraging spirals. In liquidations, collateral is used to reduce supply. Stablecoin price rises in response to imbalance with demand. This has an amplifying effect in follow-on liquidations.

THEOREM 3. Restarting the process at  $S_1$ , we have  $(\mathcal{L}_{t \wedge S_2})$  is a supermartingale and  $(Z_{t \wedge S_2})$  is a submartingale.

# [Link to Proof]

The previous result guarantees that the process, after crossing  $S_1$ , enters a deflationary regime in a precise sense. This deflationary regime can be triggered by the factors affecting  $S_1$ , such as any of the following: shocks to collateral levels, increased expectations around deleveraging costs, or depressed ETH expectations. Similarly to the results above, in real applications, these stopping times can be used by stablecoin stakeholders as indicators that the local regime is changing and to statistically estimate the probable lengths of such deleveraging spirals.

The intuition behind deleveraging spirals is illustrated in Figure 1. In an equilibrium, the stablecoin supply is matched to demand. As a first wave of speculator liquidations occur, whether voluntary deleveraging or automated by the protocol, collateral is used to repurchase the stablecoin to reduce the supply. In an imperfectly elastic market, this causes an imbalance in demand relative to supply, and an increase in stablecoin price is needed to reduce demand. This has an amplifying effect, however, in follow-on rounds of liquidations: more collateral is needed to reduce supply by the same amount because of the increased stablecoin price, and each round of liquidations continues to increase the stablecoin price.

Black Thursday in March 2020 provides strong evidence of deleveraging spirals in the Dai stablecoin. ETH price crashed ~ 50% on 12 March 2020 (Figure 2a) This triggered a wave of liquidations in Dai, as well as other cryptocurrency systems. These liquidations led to a cornering effect from deleveraging spirals in the Dai market, as shown in Figure 2b. Speculators faced premiums in excess of 10% to deleverage during the crisis and lingering premiums > 2% several weeks after. The cornering effect is also supported by lending rates on Dai, which reached high double digits during the crisis (Figure 2c). Maker was also affected by global mempool flooding on Ethereum during the crisis, which caused many Dai liquidation auctions to clear at near zero prices. This had the effect of amplifying the deleveraging effect on collateral and led to a \$4m shortfall in the system. See [3, 42] for more details. Many market participants were surprised in this crisis that



Fig. 2. Black Thursday in March 2020. (a)  $\sim 50\%$  ETH price crash (OnChainFX). (b) Deleveraging effects on Dai price and volatility (OnChainFX). (c) Deleveraging effects on Dai lending rate (LoanScan)

Dai traded at significant premiums despite the much riskier state of Maker in terms of collateral and liquidations, which our model explains as deleveraging spirals.

We now derive practical tools that will connect these regimes containing deleveraging spirals with instability in terms of forward-looking price variance of the stablecoin, and which do not require the detection of whether  $S_1$  has occurred. This formalizes the high price variation observed in Dai during and after Black Thursday. We begin in the next remark by setting up a variance estimation idea based on Taylor approximation.

REMARK 2. (Estimating variances) Taylor approximations can be applied to estimate the variances of the stablecoin process. Consider  $X_t = X_{t-1}R_t$  for return  $R_t \ge 0$ . For notational clarity, define<sup>10</sup>

$$h(\rho, n) := \arg \max_{\mathcal{L}_t} \mathbb{E}[Y_{t+1}|\mathcal{F}_t] = \mathcal{L}_t,$$

where  $\rho$ , n are realizations of  $R_t$ ,  $\bar{N}_t$ . Variance in stablecoin supply follows

$$Var(\mathcal{L}_t | \mathcal{F}_{t-1}) \approx h' \left( \mathbb{E}[R_t | \mathcal{F}_{t-1}], \bar{N}_t \right)^2 Var(R_t | \mathcal{F}_{t-1})$$

And the stablecoin price variance approximation is

$$Var(Z_t|\mathcal{F}_{t-1}) \approx \frac{\mathcal{D}h'(\mathbb{E}[R_t|\mathcal{F}_{t-1}],\bar{N}_t)^2}{\mathbb{E}[\mathcal{L}_t|\mathcal{F}_{t-1}]^4} Var(R_t|\mathcal{F}_{t-1})$$
(1)

This is given informally, but could in principle be formalized using two steps of compounded Taylor approximation error. The approximation error is arguably moderate considering that our domain is bounded away from singularities (e.g., our lower bound results on  $\mathcal{L}$ ).

This variance approximation (Eq. 1 in Remark 2) is low in the stable domain and can be high in the unstable domain, as formalized in the following Theorem 4. We introduce a few more assumptions that we use only in deriving the remaining results in this section. Note that all of these assumptions come down to assumed properties of the  $R_t$  distribution.

ASSUMPTION 12. The post-decision collateral constraint at time t is not binding in the speculator's maximization.

This first assumption means that the speculator's objective fully accounts for the post-decision collateral constraint (i.e., by maximizing the objective, the speculator by extension also satisfies the constraint), which is reasonable unless expected returns are excessively high.

# Assumption 13. Returns $R_{t-1}$ and $R_t$ are independent.

<sup>&</sup>lt;sup>10</sup>As in the case of  $\psi$ , *h* could have a subscript *t* (or equivalently other time *t* inputs), but we relax notation as we only use in the context of time *t*.

Assumption 14.  $\psi$  is twice continuously differentiable.

This last assumption restricts the density  $g_t$ . We now present the result, which applies the implicit function theorem to derive the derivatives of h, which describe the sensitivity of h to price and collateral level.

THEOREM 4. Under the above assumptions, the following hold:

- (1)  $\frac{\partial}{\partial \rho} h(\rho, n) \frac{\partial}{\partial n} h(\rho, n)$  exist.
- (2)  $\frac{\partial}{\partial \rho}h(\rho, n) \ge 0$  and is increasing in  $-\rho$  by order of  $\frac{1}{\rho}$  for  $\rho \ge \frac{b_{t-1}}{X_{t-1}}$ ,  $\mathcal{L}_t > 8$ .
- (3)  $\frac{\partial}{\partial n}h(\rho,n) \ge 0$  and is increasing in -n by order of  $\frac{1}{n}$  for  $n \ge \frac{b_{t-1}}{X_t}$ ,  $\mathcal{L}_t > 8$ .
- (4)  $\exists \varepsilon$  with  $0 < \varepsilon < 1$ , s.t.  $\frac{\partial}{\partial \rho} h(\rho, n) > 1$  if  $\rho < \varepsilon$ ,  $L_t > \frac{27}{46} \alpha \mathcal{D}$ , and  $c_t > 2$ .

As a result, the variance approximation in Eq. 1 increases by order of  $\frac{1}{R_{\star}^2}$  in  $-R_t$  and  $\frac{1}{N_t^2}$  in  $-\bar{N}_t$ .

# [Link to Proof]

Theorem 4 shows that the variance approximation in Eq. 1 in Remark 2 increases by order of  $\frac{1}{R_t^2}$  during an ETH return shock (result 2). Recall that  $R_t$  is multiplicative return, and so the effect is large for a significant shock  $R_t < 1$ . Similarly, settings with lower collateralization in the initial conditions have higher variance approximation by order of  $\frac{1}{N_t^2}$  (result 3). Such differences in initial conditions of collateral could result from, for example, different realizations of liquidations or the speculator abandoning its collateral position (and so extracting any excess collateral it can). Result 4 shows that there are cases where the h' factor in the variance approximation is > 1, meaning that the variance of  $R_t$ , the inherently volatile process, will carry through directly to  $Z_t$ , the 'stable' process.

Note that the extra conditions on the scale of  $\mathcal{L}_t$  and  $c_t$  in Theorem 4 results 2-4 may seem strange at first sight. Since the  $(Z_t)$  process is scale-invariant, as proven in Prop. 2, the results about  $Z_t$  variance hold more generally. In particular, recall that a term of  $\sim \frac{1}{\mathcal{L}_t}$  shows up in the variance approximation in Remark 2, which will cancel out the conditions on scale.

Up to this point, we have only been able to say things about variance estimations. We will now show that the 'stable' and 'unstable' regimes are well-interpreted in the following sense: given different initial conditions of the same process, the forward-looking stablecoin price variances are indeed distinct. If we start in the unstable regime, we will always have variance higher than if we start in the stable regime. The next result formalizes this.

THEOREM 5. In addition to the previous assumptions, suppose  $X_t \ge b(L_{t-1}) + \epsilon$  for some  $\epsilon > 0$  (the pre-decision collateral constraint is exceeded by  $\epsilon$ , which restricts the ranges of both  $X_t$  and  $\bar{N}_{t-1}$ ). Consider two possible states s and u of the stablecoin at time t that differ only in collateral amounts  $\bar{N}_{t-1}^s > N_{t-1}^u$  and evolve driven by the common price process  $(X_t)$ . Then the forward-looking price variances satisfy

$$Var(Z_t^s | \mathcal{F}_{t-1}) < Var(Z_t^u | \mathcal{F}_{t-1}).$$

## [Link to Proof]

Notice that special care should be given to the treatment of  $Z_t$  under the condition  $X_t \leq c(L_{t-1})$ , as the STBL price may no longer be well-defined without  $\zeta > 0$  as no collateral remains. In a real system, this is equivalent to the event that all speculators are wiped out. The reason for our condition on  $X_t$  in the above result is partly to keep things well-defined and partly because there can be a non-smooth point in h at  $X_t = b(L_{t-1})$ .

Similar variance difference results can be derived for varying initial conditions of  $X_{t-1}$  and  $\mathcal{L}_{t-1}$  as opposed to  $\bar{N}_{t-1}$ . In some sense, these are all similar as they change the initial collateralization level, though there will be some difference in price effect.

These analytical results describe regimes in which the stablecoin can be interpreted as stable and unstable. As we've discussed, they can be adapted into data-driven risk tools, for instance to estimate probabilities of peg deviations and to infer about how likely regimes are to change in the near future.

While these results apply over limited steps ahead–e.g., forward-looking variance is derived for the next time period–they *do* point in the right direction that stability domains are related to traditional measures in finance. Naturally, it would be good to have results describing further periods into the future. In principle, these could be estimated, although the process in this section is already complex. The fact that we are able to relate these regimes analytically to forward-looking variance is already a step ahead, and a valuable new result in its own right. We conjecture that it could work similarly over multi-steps, though in less tractable ways.

## 5 STABILITY IN 'PERFECT' SETTINGS

In the previous section, we considered the given model of a single speculator facing imperfectly elastic demand for STBL. We now consider idealized settings, in which STBL demand is perfectly elastic and/or unlimited speculator supply exists. In these idealized settings, we demonstrate that stablecoin can be interpreted as well-stabilized.

#### 5.1 Perfectly elastic demand

Under perfectly elastic demand, STBL demand is time-dependent  $\mathcal{D}_t$ , which adapts in each time period to match STBL supply. This results in  $Z_t = 1$ . In this case, the speculator's issue and repurchase price is always \$1 and \$ $\alpha$  in a liquidation. The problem simplifies to evaluating

$$\mathbb{E}[Y_{t+1}|\mathcal{F}_t] = \Delta_t \mathbb{E}[R_{t+1}|\mathcal{F}_t] + \int_{\frac{c_t}{X_t}}^{\infty} (\bar{N}_t X_t z - \mathcal{L}_t) g(z) dz + (1-\alpha) \int_{\frac{c_t}{X_t}}^{\frac{b_t}{X_t}} (3\mathcal{L}_t - 2\bar{N}_t X_t z) g(z) dz,$$

where the liquidation effect is now  $\ell_{t+1}(1-\alpha)$  where  $\ell_{t+1} = 3L_t - 2\bar{N}_t X_{t+1}$  and

$$c(\mathcal{L}_t) = \frac{3\alpha \mathcal{L}_t}{\bar{N}_t (2\alpha + 1)}.$$

In this setting, we have  $\frac{\partial \psi}{\partial \mathcal{L}_t} = \mathbb{E}[R_{t+1}|\mathcal{F}_t] - \mathbb{P}(A_t \cup B_t) - 3(\alpha - 1)\mathbb{P}(B_t)$ . Recalling  $\mathbb{P}(A_t)$  and  $\mathbb{P}(B_t)$  are functions of  $\mathcal{L}_t$  and supposing a non-binding collateral constraint, the speculator chooses  $\mathcal{L}_t$  such that

$$\mathbb{E}[R_{t+1}|\mathcal{F}_t] = \mathbb{P}(A_t \cup B_t) + 3(\alpha - 1)\mathbb{P}(B_t).$$

Noting that  $\mathbb{E}[R_{t+1}] \ge 1$ ,  $\mathbb{P}(A_t \cup B_t)$  is decreasing in  $\mathcal{L}_t$  but generally ~ 1, and  $\mathbb{P}(B_t)$  is increasing in  $\mathcal{L}_t$ , this is interpretable as the speculator balancing expected return against 3× the expected (constant) liquidation cost in deciding whether to issue a new unit of STBL.

In this setting, the STBL price is identically \$1 and the speculator only faces the risk of leveraged ETH declines subject to a fixed liquidation fee. Liquidations generally work well to keep the system over-collateralized, and the only real risk to STBL holders is from extreme single period declines in ETH price.

#### 5.2 Unlimited speculator supply

Suppose there is an infinite depth of speculators (with capital) ready to enter the STBL market given what they see as a profitable opportunity subject to STBL demand  $\mathcal{D}$ . A marginal speculator in such

a market would choose to deposit collateral and issue new STBL at time *t* if  $\frac{\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}_t^2} \mathbb{E}[R_{t+1}|\mathcal{F}_t] - \gamma > 0$ , where  $\gamma$  represents the marginal speculator's expected liability and liquidation cost after entering the market. Arguably,  $\gamma \sim 1$  as, in an infinite depth market, the marginal speculator can start from a position of low leverage.

The marginal profitability will be 0, which yields

$$\mathcal{L}_t = \sqrt{\gamma \mathcal{D} \mathcal{L}_{t-1} \mathbb{E}[R_{t+1} | \mathcal{F}_t]}.$$

Notice the similarity with the upper bound in Prop. 4. In this case, we attain the upper bound on supply because either the initial speculators act to increase supply or a marginal speculator will see a profitable opportunity and bring us to the upper bound.

Further using that  $(X_t)$  is a submartingale, in which case  $\mathbb{E}[R_{t+1}|\mathcal{F}_t] \ge 1$ , we find the STBL price is constrained to a small range of  $Z_0 \ge Z_t \ge \frac{1}{\gamma r}$ . This resembles the perfectly elastic demand case as existing speculators are able to liquidate positions without influencing STBL price, in this case because new marginal speculators are always willing to issue new STBL to offset a liquidation.

### 5.3 No stable region if $(X_t)$ is not a submartingale

Notice that the mechanisms that make the idealized settings well-stabilized break down when the ETH price process  $(X_t)$  is not a submartingale. This stresses how fragile the stablecoin market is to negative expectations in the primary ETH market, even under these idealized settings. In the unlimited speculator case, marginal speculators no longer enter the market if expectations are negative, and so we don't achieve the supply bound developed above. Instead, we return to the main setting of the paper, which can be interpreted as unstable under negative expectations as it leads to deleveraging effects. In the perfectly elastic demand setting, the STBL supply goes to zero as the speculator chooses not to participate.

#### 6 **DISCUSSION**

This paper presents a new stochastic model of over-collateralized stablecoins with an endogenous price. In this model, we formally characterize domains that can be interpreted as stable and unstable for the stablecoin. We prove that the stablecoin behaves in a stable way by bounding the probabilities of large deviations and quadratic variation, restricted to a certain region, and that price variance is distinctly greater in a separate region, which can be triggered by large deviations, collapsed expectations, and liquidity problems from deleveraging. We also characterize a deflationary deleveraging spiral as a submartingale, which can exacerbate liquidity problems in a crisis. These deleveraging spirals, which resemble short squeezes, are counterintuitive as they lead to stablecoin price appreciation during times of shock, whereas we might otherwise expect prices to depreciate given the riskier state of the system. Further, this appreciation is detrimental: it leads to faster collateral drawdown, and potentially shortfalls, as more collateral is required to fulfill liquidations and is accompanied by higher price variance.

An observation from the model is that the speculator chooses a collateral level *above* the required collateral factor. This is because the expected liquidation cost is greater than the \$1 face value. The speculator will desire to increase the collateralization during times when the expected liquidation cost is higher, which can occur after a shock to collateral value or if the speculator expects the collateral to be more volatile. This generally explains the high level of over-collateralization seen in Dai, which typically ranges  $2.5 - 5 \times$  although the collateral factor is  $1.5 \times$ .

The presence of deleveraging effects poses fundamental trade-offs in decentralized design. One way to bring the stablecoin closer to the 'perfect' stability cases is to increase elasticity of demand. This relies on the presence of good uncorrelated alternatives to the stablecoin. As all non-custodial

stablecoins likely face similar deleveraging risks, greater elasticity relies on custodial stablecoins or greater exchangeability to fiat currencies. Another way to bring the stablecoin closer 'perfect' stability is to increase the supply of marginal speculators. As there will not be unlimited supply of speculators with positive ETH expectations (especially during an extended bear market), this relies on having another uncorrelated collateral asset. As all decentralized assets are very correlated, this again largely relies on including custodial collateral assets, like Maker's recent addition of USDC.<sup>11</sup> While these measures strengthen the stability results, it's at the expense of greater centralization and moves the system away from being 'non-custodial'.

We suggest a way to improve the design of Dai's savings pool toward damping deleveraging effects without greater centralization through incentivizing exchangeability of Dai during deleveraging events. In its current state, the Maker system charges fees to speculators, part of which it passes on to Dai holders as an interest rate if the holder locks the Dai into a savings pool. With modified mechanics, this savings pool can provide a buffer to deleveraging effects. For instance, if we allow Dai in the savings pool to be bought out at a reasonable premium to face value by a speculator who uses it to deleverage, then deleveraging effects are bounded by the premium amount up to the size of the savings buffer. The Dai holders who participate in this savings pool are then compensated for providing a repurchase option to the speculator. The Dai holder could elect to have the repurchase fulfilled in the collateral asset, or something else, like a custodial stablecoin. In this way, this mechanism can provide some of the benefits of the 'perfect' stability settings while enabling Dai holders to choose how decentralized they want to be. A Dai holder who does not require high decentralization would elect to receive the compensation from the savings pool. Our model can be extended to consider mechanisms like this.

Since the release of our paper, mechanisms resembling this, which try to boost liquidity around liquidations to quell deleveraging spirals, have been adopted by projects such as Liquity [25]. Maker has chosen to go a different direction by maintaining direct exchangeability with the custodial USDC [28]. The stablecoin Rai has chosen a third path of instituting negative rates on stablecoin holders during crises [35].

Our model and results can also apply more broadly to synthetic and cross-chain assets and overcollateralized lending protocols that allow borrowing of illiquid and/or inelastic assets– whenever the mechanism is based on leveraged positions and leads to an endogenous price of the created or borrowed asset. Synthetic assets generally use a similar mechanism just with a different target peg. Cross-chain assets that port an asset from a blockchain without smart contract capability (e.g., Bitcoin) to a blockchain with smart contracts (e.g., Ethereum) also tend to rely on a similar mechanism. In non-custodial constructions such as [44] and [41], vault operators are required to lock ETH collateral in addition to the deliverable BTC asset. They bear a leveraged ETH/BTC exchange rate risk and face similar deleveraging risk. In particular, to reduce exposure, they need to repurchase the version of the cross-chain asset on Ethereum.

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#### REFERENCES

 Guillermo Angeris, Hsien-Tang Kao, Rei Chiang, Charlie Noyes, and Tarun Chitra. 2020. An analysis of Uniswap markets. Crypto Economic Systems 2020 (2020).

<sup>&</sup>lt;sup>11</sup>Recall that custodial assets face their own risks, however, which may not be uncorrelated in extreme crises. This includes counterparty risk, bank run risks, asset seizure risk, and effects from negative interest rates.

- Bruno Biais, Christophe Bisière, Matthieu Bouvard, and Catherine Casamatta. 2019. The Blockchain Folk Theorem. Review of Financial Studies 32, 5 (2019), 1662–1715. https://doi.org/10.1093/rfs/hhy095
- Blocknative. 2020. Evidence of Mempool Manipulation on Black Thursday: Hammerbots, Mempool Compression, and Spontaneous Stuck Transactions. https://www.blocknative.com/blog/mempool-forensics
- [4] Stephen Boyd and Lieven Vandenberghe. 2009. Convex optimization. Cambridge university press.
- [5] Dirk Bullmann, Jonas Klemm, and Andrea Pinna. 2019. In search for stability in crypto-assets: Are stablecoins the solution? ECB Occasional Paper 230 (2019).
- [6] Donald L Burkholder. 1973. Distribution function inequalities for martingales. the Annals of Probability (1973), 19-42.
- [7] Y Cao, M Dai, S Kou, L Li, and C Yang. 2018. Designing stable coins. Duo Network Whitepaper, https://duo.network/ papers/duo\_academic\_white\_paper.pdf (2018).
- [8] Tarun Chitra. 2020. Competitive equilibria between staking and on-chain lending. Crypto Economic Systems 2020 (2020).
- [9] cLabs. 2019. An analysis of the stability characteristics of Celo. Technical Report. https://celo.org/papers/Celo\_Stability\_ Analysis.pdf.
- [10] Coindesk. 17 Mar. 2020. MakerDAO adds USDC as DeFi collateral following 'Black Thursday' chaos. https://www. coindesk.com/makerdao-adds-usdc-as-defi-collateral-following-black-thursday-chaos.
- [11] C. Detrio. 2015. Smart markets for stablecoins. Technical Report. http://cdetr.io/smart-markets/.
- [12] Douglas W Diamond and Philip H Dybvig. 1983. Bank runs, deposit insurance, and liquidity. Journal of political economy 91, 3 (1983), 401–419.
- [13] Philip H Dybvig and Jaime F Zender. 1991. Capital structure and dividend irrelevance with asymmetric information. The Review of Financial Studies 4, 1 (1991), 201–219.
- [14] Alex Evans. 2019. A Ratings-Based Model for Credit Events in MakerDAO.
- [15] Lewis Gudgeon, Daniel Perez, Dominik Harz, Arthur Gervais, and Benjamin Livshits. 2020. The Decentralized Financial Crisis: Attacking DeFi. arXiv preprint arXiv:2002.08099 (2020).
- [16] Bernardo Guimaraes and Stephen Morris. 2007. Risk and wealth in a model of self-fulfilling currency attacks. Journal of Monetary Economics 54, 8 (2007), 2205–2230.
- [17] Dominik Harz, Lewis Gudgeon, Arthur Gervais, and William J. Knottenbelt. 2019. Balance: Dynamic Adjustment of Cryptocurrency Deposits. In Proceedings of the 2019 ACM SIGSAC Conference on Computer and Communications Security (CCS '19). ACM. https://eprint.iacr.org/2019/675.pdf
- [18] Hsien-Tang Kao, Tarun Chitra, Rei Chiang, and John Morrow. 2020. An Analysis of the Market Risk to Participants in the Compound Protocol. (2020).
- [19] Ariah Klages-Mundt. 14 Dec 2018. The state of stablecoins-update 2018. https://medium.com/coinmonks/the-state-of-stablecoins-update-2018-56fb82efe6de.
- [20] Ariah Klages-Mundt. Nov. 14, 2019. Vulnerabilities in Maker: oracle-governance attacks, attack DAOs, and (de)centralization. https://link.medium.com/VZG64fhmr6
- [21] Ariah Klages-Mundt, Dominik Harz, Lewis Gudgeon, Jun-You Liu, and Andreea Minca. 2020. Stablecoins 2.0: Economic Foundations and Risk-based Models. In Proceedings of the 2nd ACM Conference on Advances in Financial Technologies. 59–79.
- [22] Ariah Klages-Mundt and Andreea Minca. 2019. (In) Stability for the Blockchain: Deleveraging Spirals and Stablecoin Attacks. arXiv preprint arXiv:1906.02152 (2019).
- [23] Leland Lee and Ariah Klages-Mundt. Apr. 23, 2021. Governance Extractable Value. https://ournetwork.substack.com/ p/our-network-deep-dive-2
- [24] Alex Lipton, Thomas Hardjono, and Alex Pentland. 2018. Digital trade coin: towards a more stable digital currency. *Royal Society open science* 5, 7 (2018), 180155.
- [25] Liquity. 2020. Stability Pool and Liquidations. https://docs.liquity.org/faq/stability-pool-and-liquidations.
- [26] MakerDAO. 12 Mar 2020. Black Thursday response thread. https://forum.makerdao.com/t/black-thursday-responsethread/1433.
- [27] MakerDAO. 2017. The Dai stablecoin system whitepaper. https://makerdao.com/whitepaper/DaiDec17WP.pdf.
- [28] MakerDAO. Nov. 2020. MIP29 peg stability module. https://forum.makerdao.com/t/mip29-peg-stability-module/5071.
- [29] Stephen Morris and Hyun Song Shin. 1998. Unique equilibrium in a model of self-fulfilling currency attacks. American Economic Review (1998), 587–597.
- [30] Maureen O'Hara. 1997. Market microstructure theory. Wiley.
- [31] Cecilia Parlatore. 2016. Fragility in money market funds: Sponsor support and regulation. *Journal of Financial Economics* 121, 3 (2016), 595–623.
- [32] PeckShield. Feb. 2020. bZx Hack Full Disclosure (With Detailed Profit Analysis). https://link.medium.com/LIXArFK7e7.
- [33] PeckShield. Feb. 2020. bZx Hack II Full Disclosure (With Detailed Profit Analysis). https://link.medium.com/ 9K9LrFQ7e7.

- [34] Nicholas Platias and Marco DiMaggio. 2019. Terra money: stability stress test. Technical Report. https://agora.terra. money/t/stability-stress-test/55.
- [35] Reflexer. 2020. Rai. https://reflexer.finance/.
- [36] J. Rennison, P. Stafford, C. Smith, and R. Wigglesworth. Mar. 23, 2020. 'Great liquidity crisis' grips system as banks step back. Financial Times.
- [37] Terra Research. Jul. 2019. Increasing Robustness of the Terra Oracle. https://agora.terra.money/t/increasing-robustnessof-the-terra-oracle/82.
- [38] Chuen-Teck See and Jeremy Chen. 2008. Inequalities on the variances of convex functions of random variables. Journal of inequalities in pure and applied mathematics 9, 3 (2008), 1–5.
- [39] Synthetix. 16 Sep. 2019. Addressing claims of deleted balances. https://blog.synthetix.io/addressing-claims-of-deletedbalances/.
- [40] Synthetix. Jun. 2019. Synthetix Response to Oracle Incident. https://blog.synthetix.io/response-to-oracle-incident/.
- [41] Synthetix. Mar. 2020. tBTC: a decentralized redeemable BTC-backed ERC-20 token. https://docs.keep.network/tbtc.
- [42] Frank Topbottom. 2020. Black Thursday for MakerDAO: \$8.32 million was liquidated for 0 DAI. https://medium.com/ @whiterabbit\_hq/black-thursday-for-makerdao-8-32-million-was-liquidated-for-0-dai-36b83cac56b6
- [43] Sam M Werner, Daniel Perez, Lewis Gudgeon, Ariah Klages-Mundt, Dominik Harz, and William J Knottenbelt. 2021. SoK: Decentralized Finance (DeFi). arXiv preprint arXiv:2101.08778 (2021).
- [44] Alexei Zamyatin, Dominik Harz, Joshua Lind, Panayiotis Panayiotou, Arthur Gervais, and William J. Knottenbelt. 2019. XCLAIM: Trustless, Interoperable, Cryptocurrency-Backed Assets. In Proceedings of the IEEE Symposium on Security & Privacy, May 2019. 1254–1271. https://doi.org/10.1109/SP.2019.00085

# A EXTENDING THE MODEL TO GENERALIZED SETTINGS

We now explore how the analytical results will extend to generalized design and market settings. We can generalize the single speculator model while retaining similar analytical results. We briefly sketch out what these generalized models can look like and discuss how these relate to realistic settings.

*Concurrent collateral.* In the main analysis, we considered the speculator's collateral at stake at time *t* to be  $N_{t-1}$  (minus any applicable liquidation). Alternatively, we now consider the collateral at stake at time *t* to be

$$N_t = N_{t-1} + (\mathcal{L}_t - \mathcal{L}_{t-1})\frac{\mathcal{D}}{\mathcal{L}_t}\frac{1}{X_t}$$

where the second term represents the value in ETH obtained from issuing new stablecoins at time *t* (negative if redeeming).

For notational simplicity drop subscripts as follows:  $N_t \mapsto N$ ,  $X_t \mapsto X$ ,  $\mathcal{L}_t \mapsto \mathcal{L}$ ,  $c(L_t) \mapsto c$ ,  $b(L_t) \mapsto b$ ,  $R_{t+1} \mapsto R$ . And define  $\psi := \mathbb{E}[Y_{t+1}|\mathcal{F}_t]$ . Notice that N is now a function of  $\mathcal{L}$ . Both b and c, which were previously functions of  $\mathcal{L}$  with parameter N, are now further complicated by N's dependency on  $\mathcal{L}$ . In this setting we have

$$\psi(\mathcal{L}) = \int_{c/X}^{\infty} (NXz - \mathcal{L})g(z)dz + \int_{c/X}^{b/X} \left(3\mathcal{L} - \frac{\alpha \mathcal{D}\mathcal{L}}{2NXz - \mathcal{L}} - 2NXz\right)g(z)dz$$

The partials become more complicated algebraically since N is now a function of  $\mathcal{L}$ , e.g., partial with respect to  $\mathcal{L}$  is

$$\begin{split} \frac{\partial \psi}{\partial \mathcal{L}} &= \int_{c/X}^{\infty} \left( \frac{\mathcal{D}\mathcal{L}_{t-1}z}{\mathcal{L}^2} - 1 \right) g(z) dz \\ &+ \int_{c/X}^{b/X} \left( \frac{\alpha \mathcal{D}\mathcal{L} \left( \frac{2\mathcal{D}\mathcal{L}_{t-1}z}{\mathcal{L}^2} - 1 \right)}{(2NXz - \mathcal{L})^2} - \frac{\alpha \mathcal{D}}{2NXz - \mathcal{L}} - \frac{2\mathcal{D}\mathcal{L}_{t-1}z}{\mathcal{L}^2} + 3 \right) g(z) dz. \end{split}$$

Second partials of  $\psi$  are further complicated, as are the partials of *b* and *c*. Similar results can in principle be derived in this setting, although we need further conditions, for instance to extend the  $\psi$  concavity proof (e.g., to ensure  $\frac{\partial b}{\partial \mathcal{L}} \ge 0$ ).

Generalized STBL demand. We can consider more general STBL demand functions that depend on  $Z_t$ . For instance, consider a constant elasticity market. Let q be the quantity demanded of STBL at \$1 price and suppose the quantity demanded changes with price subject to a constant price elasticity  $-\gamma < 0$ . Then we can consider the STBL demand function

$$Q(Z_t) = q(1 - \gamma(1 - Z_t)).$$

This leads to a dollar-denominated demand function

$$\mathcal{D}(Z_t) = Z_t Q(Z_t) = Z_t q(1 - \gamma(1 - Z_t)).$$

Our analysis in the previous section becomes the simplified case where  $\gamma = 1$ , in which case dollar-denominated demand (but not quantity demanded) is constant.

In clearing the market, the generalized price process becomes

$$Z_t = \frac{1}{\gamma} \left( \frac{\mathcal{L}_t}{q} - 1 \right) + 1$$

and the collateral process becomes

$$N_t = N_{t-1} + \left(\mathcal{L}_t - \mathcal{L}_{t-1}\right) \left(\frac{1}{\gamma} \left(\frac{\mathcal{L}_t}{q} - 1\right) + 1\right) \frac{1}{X_t}$$

which we can apply to get the generalized version of  $\psi$ . The general methods used above can again apply to this formulation, though additional assumptions may again be needed, for instance to extend the  $\psi$  concavity proof.

Generalized collateral factors and supply depth. For a simple exposition, we made the simplifications that the STBL supply is composed solely of coins issued by the speculator (i.e.,  $\mathcal{L}_t = L_t$  with outside supply  $\zeta = 0$ ) and the collateral factor  $\beta = 3/2$ . The results will apply for more general  $\zeta \ge 0$  and  $\beta > 1$ . The equations for this setting are as follows.

Drop subscripts:  $\overline{N}_t \mapsto N, X_t \mapsto X, L_t \mapsto L, c_t \mapsto c, b_t \mapsto b, g_t \mapsto g, R_{t+1} \mapsto R$ . And recall  $\mathcal{L}_t = \zeta + L_t$ . Then

$$\psi(L) = (L - L_{t-1}) \frac{\mathcal{D}}{L + \zeta} \mathbb{E}[R|\mathcal{F}_t] + \int_{c/X}^{\infty} (NXz - L)g(z)dz + \int_{c/X}^{b/X} \frac{\beta L - NXz}{\beta - 1} \left(1 - \frac{\alpha \mathcal{D}}{L + \zeta - \frac{\beta L - NXz}{\beta - 1}}\right)g(z)dz$$

with c(L) and b(L) similarly redefined.

Endogeneity of collateral prices. We can also extend the model to consider certain endogenizations of collateral prices. For instance, this would model market impact effects of large collateral liquidations and also enable modelign of stablecoins like Synthetix sUSD that have endogenous collateral (see [21]). One possible way to endogenize collateral prices is to replace  $X_{t+1} \mapsto f(X_{t+1}, \bar{N}_t, L_t)$ , where f describes the market impact of a collateral liquidation at t + 1 on t + 1 collateral price and  $X_{t+1}$  describes the (exogenous) price of collateral absent any liquidation.

We expect that the general methods used in this paper can be applied to partial equilibrium settings such as this. Naturally, this would necessitate conditions on f. Notice that the transformed

collateral price process  $(f(X_{t+1}, \bar{N}_t, L_t))_{t \ge 0}$  may no longer be a submartingale. In this case, we would need further conditions on f that ensure  $\psi$  remains concave.

*Formulating as a multi-period control problem.* So far, we've specified the speculator's decisionmaking in terms of a sequence of one-period optimization problems. However, there could be better long-term strategies. Alternatively, the speculator could strategically coordinate the sequence of decisions further into the future.

This can be formulated using an exit time for the speculator based on a random clock, possibly exponential. If this terminal time is deterministic, the problem can be formulated as a dynamic program, in which the terminal decision is the one-period optimization, intermediate decisions solve a Bellman equation conditioned on the information revealed up to that point, and random returns are independent. It is possible to extend these results to a random exit time, if that exit time is a 'nice' stopping time. For instance, [2] sets up a supermodular game, for which this works.

For this to make sense conceptually, we need to assume the speculator can cash out of its position by selling to someone else at par at the exit time. This can include a noise factor of when this is possible. We expect that this noise factor would need to be independent of the state of the system in order for the problem to be tractable. A main challenge is that this will not in reality be independent.

### A.1 Note on realistic settings and applications

Realistic settings are likely to be somewhere in between the idealized settings described in the previous subsection and the single speculator with imperfectly elastic demand setting explored in this paper. As discussed in the model setup, demand will be imperfectly elastic, at least in the short term. A reasonable inelastic setting can be set by choosing an appropriate elasticity parameter for the model.

A realistic setting will have multiple speculators, including some marginal speculators, but the depth of the speculators will not be infinite. Further complications will come when different speculators maintain positions with different leverage points and/or ETH expectations. This can lead to a sequential schedule of liquidation points at a given time throughout the system, which will be reflected in a speculator's expected liquidation costs. In particular, a given speculator will take into account price effects from the potential liquidations of other speculators' positions in addition to their own when evaluating expected liquidation costs. Additionally, expected liquidation costs will reflect expectations of marginal speculators stepping in to expand the supply. Of course, given finite depth, the speculator market can dry up. For instance, the number of people who expect positive ETH returns in an extended bear market may be quite limited.

From the perspective of data-driven applications, we would use a complementary version of the model that incorporates an estimation function that the speculator uses to estimate liquidation costs and price effect since the exact market structure is not generally known in a real setting. See [1] and [18] for examples developing agent-based models in this direction.

# **B PROOFS**

In the proofs, we often use the following elementary result

LEMMA 2. For 
$$\alpha, \mathcal{D}, L \ge 0$$
,  
 $\alpha \mathcal{D} + L \le \sqrt{\alpha^2 \mathcal{D}^2 + 4\alpha \mathcal{D}L + L^2} \le \min\left(2\alpha \mathcal{D} + L, \alpha \mathcal{D} + L + \sqrt{2\alpha \mathcal{D}L}\right)$ 

PROOF. Define  $\varepsilon := \sqrt{\alpha^2 \mathcal{D}^2 + 4\alpha \mathcal{D}L + L^2}$ . We have  $\varepsilon \leq 2\alpha \mathcal{D} + L$  as long as  $\alpha \mathcal{D} \geq L(\sqrt{3} - 2)$ , which is true since  $\alpha, \mathcal{D}, L \geq 0$ . Next, notice that  $\varepsilon = \sqrt{(\alpha \mathcal{D} + L)^2 + 2\alpha \mathcal{D}L}$ . Thus  $\varepsilon > \alpha \mathcal{D} + L$  since  $2\alpha \mathcal{D}L \geq 0$ . Lastly, by concavity,  $\varepsilon \leq \alpha \mathcal{D} + L + \sqrt{2\alpha \mathcal{D}L}$ .

*Prop.* 1.

PROOF. Consider  $X_{t+1} = X_t R_{t+1}$ . For notational simplicity, drop subscripts as follows:  $\bar{N}_t \mapsto N$ ,  $X_t \mapsto X$ ,  $\mathcal{L}_t \mapsto \mathcal{L}$ ,  $\Delta = \mathcal{L}_t - \mathcal{L}_{t-1}$ ,  $c(L_t) \mapsto c$ ,  $b(L_t) \mapsto b$ ,  $g_t \mapsto g$ ,  $R_{t+1} \mapsto R$ . Define  $\psi := \mathbb{E}[Y_{t+1}|\mathcal{F}_t]$ . Then

$$\psi(\mathcal{L}) = \frac{\Delta \cdot \mathcal{D}}{\mathcal{L}} \mathbb{E}[R|\mathcal{F}_t] + \int_{c/X}^{\infty} (NXz - \mathcal{L})g(z)dz + \int_{c/X}^{b/X} \left(3\mathcal{L} - \frac{\alpha \mathcal{D}\mathcal{L}}{2NXz - \mathcal{L}} - 2NXz\right)g(z)dz$$

Recall that the integrand factor  $\left(3\mathcal{L} - \frac{\alpha \mathcal{D}\mathcal{L}}{2NXz-\mathcal{L}} - 2NXz\right)$  evaluated at Xz = c is  $\mathcal{L} - Nc$  (the liquidation zeros out the speculator's collateral position), and evaluated at Xz = b is 0 (on the threshold of liquidation).

Taking derivatives using Leibniz integral rule:

$$\begin{split} \frac{\partial \psi}{\partial \mathcal{L}} &= \frac{\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}^2} \mathbb{E}[R|\mathcal{F}_t] - \left(NX\frac{c}{X} - \mathcal{L}\right)g\left(\frac{c}{X}\right)\frac{\partial c}{\partial \mathcal{L}}\frac{1}{X} - \int_{\frac{c}{X}}^{\infty}g(z)dz \\ &- \left(\mathcal{L} - NX\frac{c}{X}\right)g\left(\frac{c}{X}\right)\frac{\partial c}{\partial \mathcal{L}}\frac{1}{X} + \int_{\frac{c}{X}}^{\frac{b}{X}} \left(3 - \frac{\alpha \mathcal{D}NXz}{2(NXz - \mathcal{L})^2}\right)g(z)dz \\ &= \frac{\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}^2} \mathbb{E}[R|\mathcal{F}_t] - \int_{\frac{c}{X}}^{\infty}g(z)dz + \int_{\frac{c}{X}}^{\frac{b}{X}} \left(3 - \frac{\alpha \mathcal{D}NXz}{2(NXz - \mathcal{L})^2}\right)g(z)dz \\ &\frac{\partial^2 \psi}{\partial \mathcal{L}^2} = -\frac{2\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}^3} \mathbb{E}[R|\mathcal{F}_t] + g\left(\frac{b}{X}\right)\frac{\partial b}{\partial L}\frac{1}{X}\left(3 - \frac{\alpha \mathcal{D}Nb}{2(Nb - \mathcal{L})^2}\right) \\ &- g\left(\frac{c}{X}\right)\frac{\partial c}{\partial L}\frac{1}{X}\left(2 - \frac{\alpha \mathcal{D}Nc}{2(Nc - \mathcal{L})^2}\right) - \int_{\frac{c}{X}}^{\frac{b}{X}} \frac{\alpha \mathcal{D}NXz}{(NXz - \mathcal{L})^3}g(z)dz \end{split}$$

Notice that  $\frac{\partial b}{\partial L} > 0$ ,  $\frac{\partial c}{\partial L} > 0$ ,  $g \ge 0$ , and

$$3 - \frac{\alpha \mathcal{D}Nb}{2(Nb - \mathcal{L})^2} = 3 - \frac{\alpha \mathcal{D}\beta \mathcal{L}}{2(\mathcal{L}(\beta - 1))^2} = 3 - \frac{3\alpha \mathcal{D}}{\mathcal{L}} < 0$$

by assumption that liquidation repurchase price always  $\geq 1$ . Additionally, the remaining integral is always positive as the integrand is positive between the limits and and  $g \geq 0$ . Finally,  $\mathbb{E}[R|\mathcal{F}_t] \geq 0$  since  $(X_t)$  is a submartingale. Thus under the given conditions,  $\frac{\partial^2 \psi}{\partial \mathcal{L}^2} \leq 0$  as all terms are  $\leq 0$ .

Further supposing that either  $\mathbb{E}[R|\mathcal{F}_t] > 0$  or  $\mathbb{P}\left(c(L) < XR < b(L)\right) = \int_{c/X}^{b/X} g(z)dz > 0$ , then  $\frac{\partial^2 \psi}{\partial \mathcal{L}^2} < 0.$ 

Notice that the  $\frac{1}{2}$  in the bound is related to the choice  $\beta = \frac{3}{2}$ .

# Prop. 2.

**PROOF.** Easily verifiable by substitution, noting that factors of  $\gamma$  cancel in the integral limits.  $\Box$ 

Prop. 3.

**PROOF.** The speculator can at most buy back using all its ETH. At time *t*, this amount is the solution  $\Delta_t$  to the following

$$\frac{\Delta_t \mathcal{D}}{L_{t-1} + \Delta_t} + N_{t-1} X_t - L_{t-1} - \Delta_t = 0$$

supposing there is no liquidation at time *t*. It is straightforward to verify the solution, giving the lower bound:

$$\Delta_t \geq \frac{1}{2} \Big( -\sqrt{\mathcal{D}^2 - 4\mathcal{D}\mathcal{L}_{t-1} + 2\mathcal{D}N_{t-1}X_t + N_{t-1}^2 X_t^2} + \mathcal{D} - 2\mathcal{L}_{t-1} + N_{t-1}X_t \Big).$$

Note that if the speculator is not soluble at time *t*, then there is no real solution.

#### Prop. 4.

PROOF. As above, consider  $X_{t+1} = X_t R_{t+1}$ . And for notational simplicity, drop subscripts as follows:  $\overline{N}_t \mapsto N, X_t \mapsto X, \mathcal{L}_t \mapsto \mathcal{L}, \Delta = \mathcal{L}_t - \mathcal{L}_{t-1}, c(L_t) \mapsto c, b(L_t) \mapsto b, g_t \mapsto g, R_{t+1} \mapsto R, \mathbb{P}(A_t | \mathcal{F}_t) \mapsto \mathbb{P}(A), \mathbb{P}(B_t | \mathcal{F}_t) \mapsto \mathbb{P}(B).$ 

Suppose the first condition is true. We have

$$\begin{split} \frac{\partial \psi}{\partial \mathcal{L}} &= \frac{\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}^2} \mathbb{E}[R|\mathcal{F}_t] - \int_{\frac{c}{X}}^{\infty} g(z)dz + \int_{\frac{c}{X}}^{\frac{b}{X}} \left(3 - \frac{\alpha \mathcal{D}NXz}{2(NXz - \mathcal{L})^2}\right) g(z)dz \\ &\leq \frac{\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}^2} \mathbb{E}[R|\mathcal{F}_t] - \mathbb{P}(A \cup B) \\ &\leq \frac{\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}^2} \mathbb{E}[R|\mathcal{F}_t] - \kappa^{-1} \end{split}$$

Notice this is monotonic decreasing in  $\mathcal{L}$  over the domain, so the critical point will be a bound for the optimal value of  $\mathcal{L}^*$ . Setting equal to 0, we have

$$\mathcal{L}^* \leq \sqrt{\kappa \mathcal{D} \mathcal{L}_{t-1} \mathbb{E}[R|\mathcal{F}_t]}$$

Now suppose the second condition is true instead. We have

$$\begin{split} \frac{\partial \psi}{\partial \mathcal{L}} &= \frac{\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}^2} \mathbb{E}[R|\mathcal{F}_t] - \int_{\frac{b}{X}}^{\infty} g(z)dz + 2\int_{\frac{c}{X}}^{\frac{b}{X}} g(z)dz - \int_{\frac{c}{X}}^{\frac{b}{X}} \frac{\alpha \mathcal{D}NXz}{2(NXz - \mathcal{L})^2} g(z)dz \\ &\leq \frac{\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}^2} \mathbb{E}[R|\mathcal{F}_t] - \left(\mathbb{P}(A) - 2\mathbb{P}(B)\right) \\ &\leq \frac{\mathcal{D}\mathcal{L}_{t-1}}{\mathcal{L}^2} \mathbb{E}[R|\mathcal{F}_t] - \kappa^{-1} \end{split}$$

which delivers the desired result as above.

Prop. 5.

PROOF. By assuming  $T_{Z_0} > \tau$ , we have  $Z_0 \ge Z_{t \wedge \tau}$ . Applying Proposition 4 to  $Z_t = \frac{\mathcal{D}}{\mathcal{L}_t}$  provides  $Z_{t \wedge \tau} \ge \sqrt{\frac{\mathcal{D}}{\kappa \mathcal{L}_{t \wedge \tau-1}r}}$ . Notice that the upper bound on  $\mathcal{L}_t$  and the lower bound on  $Z_t$  can be written respectively as increasing and decreasing sequences in t starting from initial state as follows:

$$\overline{\mathcal{L}}_{t} = (\kappa \mathcal{D}r)^{\frac{2^{t}-1}{2^{t}}} \mathcal{L}_{0}^{\frac{1}{2^{t}}}$$
$$\underline{Z}_{t} = \frac{\mathcal{D}}{(\kappa \mathcal{D}r)^{\frac{2^{t}-1}{2^{t}}} \mathcal{L}_{0}^{\frac{1}{2^{t}}}}$$

These have limits  $\overline{\mathcal{L}}_{\infty} = \kappa \mathcal{D}r$  and  $\underline{Z}_{\infty} = \frac{1}{\kappa r}$  that also bound  $\mathcal{L}_t$  and  $Z_t$  respectively.

Prop. 6.

PROOF. For  $t - 1 < \tau$ ,

$$\frac{\mathcal{D}}{\mathbb{E}[\mathcal{L}_t|\mathcal{F}_{t-1}]} \leq \mathbb{E}\left[\frac{\mathcal{D}}{\mathcal{L}_t}|\mathcal{F}_{t-1}\right] \leq \frac{\mathcal{D}}{\mathcal{L}_{t-1}}$$

by Jensen's inequality and condition for  $\tau > t - 1$ . Thus we have

$$\mathbb{E}[\mathcal{L}_{t\wedge\tau}|\mathcal{F}_{t-1}] \geq \mathcal{L}_{t\wedge\tau-1}$$

and  $(\mathcal{L}_{t\wedge\tau})$  is a submartingale.  $(Z_{t\wedge\tau})$  is a supermartingale by condition of  $\tau$ .

Applying Proposition 5,  $\mathcal{L}_{t\wedge\tau}$  is bounded above and  $Z_{t\wedge\tau}$  is bounded below. Thus they converge a.s. by Doob's martingale convergence theorem.

Prop. 7.

PROOF. The first inequality follows from Prop. 5 and supermartingale properties.

Since  $Z_{t\wedge\tau}$  is supermartingale, we have  $Z_{t-1} \ge \mathbb{E}[Z_t|\mathcal{F}_{t-1}]$ . Assume  $(\mathbb{E}[R_{t+1}|\mathcal{F}_t])$  is non-decreasing for  $t < \tau$ . Then subject to the stopping time  $\tau$ ,

$$\mathbb{E}[Z_{t}|\mathcal{F}_{t-1}] \geq \mathbb{E}\left[\sqrt{\frac{\mathcal{D}}{\kappa\mathcal{L}_{t-1}\mathbb{E}[R_{t+1}|\mathcal{F}_{t}]}}|\mathcal{F}_{t-1}\right] \qquad (Apply Prop. 4)$$

$$\geq \sqrt{\frac{\mathcal{D}}{\kappa\mathcal{L}_{t-1}\mathbb{E}\left[\mathbb{E}[R_{t+1}|\mathcal{F}_{t}]|\mathcal{F}_{t-1}\right]} \qquad (Jensen's inequality)$$

$$= \sqrt{\frac{\mathcal{D}}{\kappa\mathcal{L}_{t-1}\mathbb{E}[R_{t+1}|\mathcal{F}_{t-1}]}} \qquad (Tower property)$$

$$\geq \sqrt{\frac{\mathcal{D}}{\kappa\mathcal{L}_{t-1}\mathbb{E}[R_{t}|\mathcal{F}_{t-1}]}}$$

since  $\mathbb{E}[R_{t+1}|\mathcal{F}_t] \geq \mathbb{E}[R_t|\mathcal{F}_{t-1}].$ 

Lemma 1.

PROOF. For  $t - 1 < \tau \wedge T_m$ ,

$$\mathbb{E}\left[|m - Z_t||\mathcal{F}_{t-1}\right] \ge |\mathbb{E}[m - Z_t|\mathcal{F}_{t-1}]|$$
$$\ge |m - Z_{t-1}|$$

by Jensen's inequality and the condition for  $t - 1 < T_m$  that  $m - Z_{t-1} \ge 0$ . Thus  $\left(Z'_{t \wedge \tau \wedge T_m}\right)$  is a non-negative submartingale.

Prop. 8.

PROOF. Note for  $t < \tau \wedge T_m$ , have  $Z_t^* \leq m$ , and so  $Z_{\tau \wedge T_m-1}'^* \leq m - \frac{1}{\kappa r}$ . Thus  $Z_{\tau \wedge T_m}'^* \leq \max\left(m - \frac{1}{\kappa r}, Z_{\tau \wedge T_m}'\right)$ .

Consider time  $t = \tau \wedge T_m$  and note that optional stopping applies since *Z* is bounded. Denote  $W := m - Z_t, E := \mathbb{E}[-W|Z_t > m]$ , and  $p := \mathbb{P}(Z_t \le m)$ . From optional stopping, we recall that  $m \ge \mathbb{E}[Z_t] \ge \frac{1}{\kappa r}$ , and so  $0 \le \mathbb{E}[W] \le m - \frac{1}{\kappa r}$ . Then

$$\mathbb{E}[W] = \mathbb{E}[W \, \mathbb{1}_{Z_t \le m}] - \mathbb{E}[-W \, \mathbb{1}_{Z_t > m}]$$
$$\leq p\left(m - \frac{1}{\kappa r}\right) - (1 - p)E$$

Combining with  $0 \leq \mathbb{E}[W]$ , we have  $0 \leq p(m - \frac{1}{\kappa r}) - (1 - p)E$ , which gives

$$p \ge \frac{E}{m - \frac{1}{\kappa r} + E}$$

Then noting that  $(1-p)E \leq E(1-\frac{E}{m-\frac{1}{\kappa r}+E}), p \leq 1$ , and  $\mathbb{E}[Z'_t] = \mathbb{E}[W \mathbbm{1}_{Z_t \leq m}] + \mathbb{E}[-W \mathbbm{1}_{Z_t > m}],$ we have  $\mathbb{E}[Z'^*] \leq p \mathbb{E}[Z'^* ] + (1-p)E$ 

$$\mathbb{E}[Z_t^{**}] \le p \mathbb{E}[Z_{t-1}^{**}] + (1-p)E$$
$$\le m - \frac{1}{\kappa r} + E\left(1 - \frac{E}{m - \frac{1}{\kappa r} + E}\right)$$
$$= m - \frac{1}{\kappa r} + \frac{E(m - \frac{1}{\kappa r})}{m - \frac{1}{\kappa r} + E}$$

Notice further that given either of the following conditions •  $\frac{1}{2} > m$  and  $E > \frac{1}{2} - m$ 

• 
$$\frac{1}{\kappa r} > m$$
 and  $E > \frac{1}{\kappa r} -$   
•  $\frac{1}{\kappa r} = m$  and  $E > 0$   
•  $\frac{1}{\kappa r} < m$  ad  $E \ge 0$ 

then

$$0 \le (1-p)E \le \frac{E(m-\frac{1}{\kappa r})}{m-\frac{1}{\kappa r}+E} \le m-\frac{1}{\kappa r}$$

Thus, recalling we used  $t = \tau \wedge T_m$ , we get the following result

$$\mathbb{E}[Z_{\tau\wedge T_m}'^*] \le 2\left(m - \frac{1}{\kappa r}\right)$$

Theorem 1.

PROOF. Given Lemma 1 and Prop. 8 and noting  $\mathbb{E}[Z'_{\tau \wedge T_m}] \leq \mathbb{E}[Z'^*_{\tau \wedge T_m}]$ , apply Doob's maximal inequality.

#### Theorem 2.

PROOF. Apply Theorem 3.1 in [6], noting that  $\sup_n \mathbb{E}[Z'_{n \wedge \tau \wedge T_m}] \leq \mathbb{E}[Z'^*_{\tau \wedge T_m}]$  by Jensen's inequality.  $\Box$ 

#### Theorem 3.

PROOF. For  $S_1 \leq t < S_2$ , we have

$$\mathbb{E}\left[\frac{\mathcal{D}}{\mathcal{L}_{t}}|\mathcal{F}_{t-1}\right] \geq \frac{\mathcal{D}}{\mathbb{E}[\mathcal{L}_{t}|\mathcal{F}_{t-1}]} \\ \geq \frac{\mathcal{D}}{f_{t-1}}$$

by Jensen's inequality and the  $S_1$  condition  $\mathbb{E}[\mathcal{L}_t | \mathcal{F}_{t-1}] \leq \mathcal{L}_{t-1}$ . Thus  $(Z_{S_1 \vee t \wedge S_2})$  is a submartingale (though note that it can be a submartingale for more general stopping times than this).

*L* started at  $S_1$  and stopped  $S_2$  is a supermartingale (by def).

# Theorem 4.

PROOF. As above, consider  $X_{t+1} = X_t R_{t+1}$ . And for notational simplicity, drop subscripts as follows:  $\bar{N}_t \mapsto N, X_{t-1} \mapsto X$  (notice this is different from previous usage),  $\mathcal{L}_t \mapsto \mathcal{L}, \Delta = \mathcal{L}_t - \mathcal{L}_{t-1}, c(L_t) \mapsto c, b(L_t) \mapsto b$ , and  $g_t \mapsto g$ .

Let  $\rho$  be (deterministic) variable representing the outcome of  $R_t$ , such that now we have the outcome  $X_t = X\rho$ . And define  $h(\rho) = \arg \max_L \psi(\rho, L) = \mathbb{E}[Y_{t+1}|\mathcal{F}_t]$ . By first order condition,  $\frac{\partial}{\partial L}\psi(\rho, h(\rho)) = 0$ . The assumptions on  $\psi$  provide unique maximum and fulfill conditions of the implicit function theorem, which gives us  $\frac{\partial h}{\partial \rho}(\rho)$  exists and

$$\frac{\partial h}{\partial \rho}(\rho) = -\frac{\frac{\partial^2}{\partial \rho \partial L}\psi(\rho, h(\rho))}{\frac{\partial^2}{\partial L^2}\psi(\rho, h(\rho))}$$

Calculating derivatives using the Leibniz integral rule (recalling *c*, *b* are functions of *L*),

$$\begin{split} \frac{\partial^2 \psi}{\partial \rho \partial L} &= g\left(\frac{c}{X\rho}\right) \frac{c}{X\rho^2} \left(4 - \frac{\alpha \mathcal{D}Nc}{2(Nc - \mathcal{L})^2}\right) - g\left(\frac{b}{X\rho}\right) \frac{b}{X\rho^2} \left(3 - \frac{\alpha \mathcal{D}Nb}{2(Nb - \mathcal{L})^2}\right) \\ &+ \int_{\frac{c}{X\rho}}^{\frac{b}{X\rho}} \frac{\alpha \mathcal{D}NXz(NX\rho z + \mathcal{L})}{2(NX\rho z - \mathcal{L})^3} g(z) dz \\ \frac{\partial^2 \psi}{\partial L^2} &= -\frac{2\mathcal{D}\mathcal{L}_{t-1}\mathbb{E}[R_{t+1}]}{\mathcal{L}^3} + g\left(\frac{b}{X\rho}\right) \frac{\partial b}{\partial L} \frac{1}{X\rho} \left(3 - \frac{\alpha \mathcal{D}Nb}{2(Nb - \mathcal{L})^2}\right) \\ &- g\left(\frac{c}{X\rho}\right) \frac{\partial c}{\partial L} \frac{1}{X\rho} \left(2 - \frac{\alpha \mathcal{D}Nc}{2(Nc - \mathcal{L})^2}\right) - \int_{\frac{c}{X\rho}}^{\frac{b}{X\rho}} \frac{\alpha \mathcal{D}NX\rho z}{(NX\rho z - \mathcal{L})^3} g(z) dz \end{split}$$

Notice that (and continuing with  $\beta = 3/2$ )

$$3 - \frac{\alpha \mathcal{D}Nb}{2(Nb - \mathcal{L})^2} = 3 - \frac{\alpha \mathcal{D}\beta \mathcal{L}}{2(\mathcal{L}(\beta - 1))^2} = 3 - \frac{3\alpha \mathcal{D}}{\mathcal{L}} < 0$$

by assumption that liquidation repurchase price always  $\geq$  1. And

$$\frac{\alpha \mathcal{D}Nc}{2(Nc-\mathcal{L})^2} \leq \frac{\frac{1}{2}\alpha \mathcal{D}(2\alpha \mathcal{D} + L - \alpha \mathcal{D} + L)}{-2\alpha \mathcal{D}(2\alpha \mathcal{D} + L) + 2L(\alpha \mathcal{D} + L) + 2\alpha^2 \mathcal{D}^2 + 2\alpha \mathcal{D}L + 2L^2}$$
$$= \frac{\alpha \mathcal{D}(\alpha \mathcal{D} + 2L)}{4(\alpha \mathcal{D} + L)(2L - \alpha \mathcal{D})}$$
$$= \frac{\alpha \mathcal{D}}{12(\alpha \mathcal{D} + L)} + \frac{\alpha \mathcal{D}}{3(2L - \alpha \mathcal{D})}$$
$$\leq \frac{1}{12} + \frac{\alpha \mathcal{D}}{3(2L - \alpha \mathcal{D})}$$

This is  $\leq 2$  when  $L \geq \frac{27}{46} \alpha \mathcal{D}$ . Thus under this condition

$$4 - \frac{\alpha \mathcal{D}Nc}{2(Nc - \mathcal{L})^2} > 2 - \frac{\alpha \mathcal{D}Nc}{2(Nc - \mathcal{L})^2} \ge 0$$

Note that all terms of  $\frac{\partial^2 \psi}{\partial \rho \partial L}$  are non-negative and all terms of  $\frac{\partial^2 \psi}{\partial L^2}$  are non-positive. Given  $\rho \geq b/X$ , we have  $g\left(\frac{c}{X\rho}\right)$  and  $g\left(\frac{b}{X\rho}\right)$  are increasing in  $1/\rho$ . Note also that  $\frac{\partial b}{\partial L}$ ,  $\frac{\partial c}{\partial L}$ , and  $\frac{2\mathcal{D}\mathcal{L}_{t-1}\mathbb{E}[R_{t+1}]}{\mathcal{L}^3}$  are constant in  $\rho$ . Lastly, the numerator and denominator integrals can be rewritten respectively as

$$\frac{1}{\rho} \int_{c}^{b} \frac{\alpha \mathcal{D} N z (N z + \mathcal{L})}{2 (N z - \mathcal{L})^{3}} g\left(\frac{z}{X \rho}\right) dz \text{ and } \int_{c}^{b} \frac{\alpha \mathcal{D} N z}{(N z - \mathcal{L})^{3}} g\left(\frac{z}{X \rho}\right) dz$$

and  $\frac{\alpha \mathcal{D}Nz(Nz+\mathcal{L})}{2(Nz-\mathcal{L})^3} \ge \frac{\alpha \mathcal{D}Nz}{(Nz-\mathcal{L})^3}$  given  $Nz + \mathcal{L} \ge Nc + \mathcal{L} > 2$ , for which  $\mathcal{L} > 8$  is sufficient. And so the terms in the numerator of  $|h'(\rho)|$  are growing by a factor  $1/\rho$  faster than the terms in the denominator as  $\rho$  decreases, proving (2).

Next, note that under the condition  $0 < \rho < 1$ ,

$$\frac{b}{X\rho^2} = \frac{\beta L}{NX\rho^2} = \frac{db}{dL}\frac{L}{X\rho^2} \ge \frac{db}{dL}\frac{1}{X\rho}$$
$$\frac{c}{X\rho^2} \ge \frac{dc}{dL}\frac{1}{X\rho^2} \ge \frac{dc}{dL}\frac{1}{X\rho}$$

The last relation uses the fact that  $\frac{dc}{dL} \leq \frac{2\alpha \mathcal{D} + L}{2(\alpha \mathcal{D} + L)} + 1 < 2$ , and so  $c > \frac{dc}{dL}$  under the problem setup. Next note that for  $\rho \leq \frac{\mathcal{L}}{8}$  and  $c \leq X\rho z \leq b$ , we have

$$\frac{\alpha \mathcal{D}NXz(NX\rho z + \mathcal{L})}{2(NX\rho z - \mathcal{L})^3} \geq \frac{\alpha DNX\rho z}{(NX\rho z - \mathcal{L})^3}$$

This is because the expression (1) simplifies to  $NX\rho z + \mathcal{L} \ge 2\rho$ , (2) to be true over the whole range of *z*, we need  $Nc + \mathcal{L} \ge 2\rho$ , and (3)  $\rho \le \frac{\mathcal{L}}{8}$  is sufficient for this. Thus

$$\int_{\frac{c}{X\rho}}^{\frac{b}{X\rho}} \frac{\alpha \mathcal{D}NXz(NX\rho z + \mathcal{L})}{2(NX\rho z - \mathcal{L})^3} g(z)dz \ge \int_{\frac{c}{X\rho}}^{\frac{b}{X\rho}} \frac{\alpha \mathcal{D}NX\rho z}{(NX\rho z - \mathcal{L})^3} g(z)dz$$

under these conditions.

Then note that all terms in the numerator of  $h'(\rho)$  are greater than and grow faster in  $1/\rho$  than the comparable terms in the denominator. This leaves the first term in the numerator, which is

constant in  $\rho$ . To get (3), then note that  $\varepsilon$  can be chosen such that for  $\rho = \varepsilon$ , the numerator and denominator are equal.

We can derive the results for  $\frac{\partial h}{\partial n}$  in essentially the same way. Alter the above dropping of subscripts with  $X_t \mapsto X$ , let *n* be a variable representing the realization of  $\bar{N}_t$ , and consider *h* as a function of *n*. Note the following relevant derivatives.

$$\frac{\partial b}{\partial n} = -\frac{\beta L}{n^2} = -\frac{b}{n}$$
$$\frac{\partial c}{\partial n} = -\frac{1}{2n^2} \left( \sqrt{\alpha^2 \mathcal{D}^2 + 4\alpha \mathcal{D}L + L^2} - \alpha \mathcal{D} + L \right) = -\frac{c}{n}$$
$$\frac{\partial^2 \psi}{\partial n \partial L} = g\left(\frac{c}{X}\right) \frac{c}{n} \left( 2 - \frac{\alpha \mathcal{D}nc}{2(nc - \mathcal{L})^2} \right) - g\left(\frac{b}{X}\right) \frac{b}{n} \left( 3 - \frac{\alpha \mathcal{D}nb}{2(nb - \mathcal{L})^2} \right)$$
$$+ \int_{\frac{c}{X}}^{\frac{b}{X}} \frac{\alpha \mathcal{D}nXz(nXz + \mathcal{L})}{2(nXz - \mathcal{L})^3} g(z) dz$$

And translating the following to the new notation

$$\frac{\partial^2 \psi}{\partial L^2} = -\frac{2\mathcal{D}\mathcal{L}_{t-1} \mathbb{E}[R_{t+1}]}{\mathcal{L}^3} + g\left(\frac{b}{X}\right) \frac{\partial b}{\partial L} \frac{1}{X} \left(3 - \frac{\alpha \mathcal{D}nb}{2(nb - \mathcal{L})^2}\right) \\ - g\left(\frac{c}{X}\right) \frac{\partial c}{\partial L} \frac{1}{X} \left(2 - \frac{\alpha \mathcal{D}nc}{2(nc - \mathcal{L})^2}\right) - \int_{\frac{c}{X}}^{\frac{b}{X}} \frac{\alpha \mathcal{D}nXz}{(nXz - \mathcal{L})^3} g(z) dz$$

And by applying implicit function theorem, we get

$$\frac{\partial h}{\partial n}(n) = -\frac{\frac{\partial^2}{\partial n \partial L}\psi(n, h(n))}{\frac{\partial^2}{\partial L^2}\psi(n, h(n))}$$

From here we can proceed with the same analysis using factors of  $\frac{1}{n}$  instead of  $\frac{1}{n}$ .

## Theorem 5.

**PROOF.** For notational simplicity, drop subscripts  $X_t \mapsto X$ ,  $\overline{N}_{t-1} \mapsto N$ ,  $\mathcal{L}_{t-1} \mapsto \mathcal{L}$ . And consider x a realization of X as variable in h. Define the function  $f(X, n) = \frac{1}{h(X,n)}$  where n represents the realization of N. With probability 1, the following are true:

- h is concave in x and n because h' is decreasing, as shown in the previous result.
- *f* is differentiable (wrt *n* and *x*) over domain using chain rule and implicit function theorem.
- *f* is convex: it's the composition of 1/*x* and *h*, and since 1/*x* is convex and non-increasing and *h* is concave, so is *f* (see [4] 3.2.4).
- *f* is (strictly) decreasing (in *n* and *x*) since *h* is increasing.
- By assumption, we've restricted NX. The derivative of f at the minimum value exists and is bounded.
- *f* is non-negative since *h* is non-negative.
- $\frac{\partial f}{\partial n}$  is (strictly) increasing in *n*. We have

$$f'(x,n) = -\frac{1}{h(x,n)^2}h'(x,n)$$

where h'(x, n) is derived in the previous proof using the implicit function theorem. *h* is increasing in *n* and *h'* is non-negative and decreasing in *n*. Thus  $\frac{h'}{h^2}$  is decreasing in *n*, and so  $-\frac{h'}{h^2}$  is increasing.

- $\frac{\partial h}{\partial n}$  is increasing in x. This can be seen using the formulation at the end of the proof for the previous result as terms in  $\frac{\partial^2 \psi}{\partial L^2}$  grow slower in x (in magnitude) than terms in  $\frac{\partial^2 \psi}{\partial n \partial L}$ . In particular, the first term of  $\frac{\partial^2 \psi}{\partial L^2}$  is decreasing in magnitude since L is increasing in x. And the integral in  $\frac{\partial^2 \psi}{\partial n \partial L}$  increases faster in x than the integral in  $\frac{\partial^2 \psi}{\partial L^2}$ , as can be seen by comparing the integrand numerators (a factor of  $x^2$  in  $\frac{\partial^2 \psi}{\partial n \partial L}$  vs. a factor of x in  $\frac{\partial^2 \psi}{\partial L^2}$ ).
- $\frac{\partial f}{\partial n}$  is (strictly) increasing in x This is because h is increasing in x and  $\frac{\partial h}{\partial n}$  is non-negative and increasing in x (previous bullet).

Note additionally that, from the system setup assumptions, all of the functions are appropriately bounded.

Thus we can apply Theorem 3.1 in [38] to get

$$\operatorname{Var}(f(X, N^s)|\mathcal{F}_{t-1}) < \operatorname{Var}(f(X, N^u)|\mathcal{F}_{t-1}).$$

Note that the variances exist because  $h = \mathcal{L}_t$  is bounded, as shown in previous results. The variances of  $Z_t^s$  and  $Z_t^u$  are then obtained by multiplying the above inequality by  $\mathcal{D}^2$ .